Financial Integration and Systemic Risk*

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September 16, 2005

Abstract

Recent empirical studies criticize the sluggish financial integration in the euro area and find that only interbank money markets are fully integrated so far. This paper studies the optimal regional and/or sectoral integration of financial systems given that integration is restricted to the interbank market. Based on Allen and Gale (2000)'s seminal analysis of financial contagion we derive the interbank market structure that maximizes consumers' ex-ante expected utility, i.e. that optimizes the trade-off between the contagion and the diversification effect. We analyze the impact of various structural parameters including the underlying stochastic structure on this trade-off. In addition we derive the efficient design of the interbank market that allows for a cross-regional risk sharing between banks. We also provide a measure for the efficiency losses that result if financial integration is limited to an integration of the interbank market.

Keywords: Interbank Market, Risk Sharing, Financial Contagion, Financial Integration

JEL Classification: D61; E44; G10; G21

*The views expressed here are those of the authors and not necessarily those of the Deutsche Bundesbank. We thank seminar participants at Deutsche Bundesbank and in particular Phil Dybvig, Heinz Herrmann, and Julia von Borstel for helpful comments and suggestions.
1 Introduction

The most integrated financial market in the Euro area is the interbank or money market and within the money market the most integrated segment is the unsecured interbank market.\(^1\) Besides its importance for a smooth monetary policy transmission across the member states a high degree of integration of this market is also beneficial under risk sharing considerations. An interlinked banking system within the Euro area provides an insurance mechanism against regional liquidity shocks. However, the flip side of the coin is that interbank claims can bring about a risk of financial contagion, i.e. increase the systemic risk. The liquidity crisis of a single institute can easily spill-over to other banks if those financial intermediaries hold interbank deposits with the troubled bank.

In the present paper we evaluate the trade-off that these two effects bring about and analyze under which conditions the benefits of a cross-regional risk sharing dominate the expected costs of financial contagion. Doing so we also take into account the optimal institutional arrangement of the interbank market. In particular, we derive how the interbank market has to be designed in order to provide a means for a constraint efficient financial integration. Interestingly, for reasonable cross-regional distributions of liquidity shocks we find that an efficient arrangement has to allow for the possibility of financial contagion. However, we also show that simple interbank arrangements do not allow to capture all benefits from a cross regional risk sharing. More complex arrangements, i.e. cross-country mergers, of financial institutions can improve the cross-country insurance against regional liquidity shocks.

Our paper is closely related to the seminal work by Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000). Like our paper both show that regional-specific liquidity shocks provide a rationale for interlinking regionally restricted banks of a Diamond and Dybvig (1983)-type. Interbank deposits in the case of Allen and Gale (2000) and interbank lines of credit in the case of Freixas, Parigi, and Rochet (2000) enable banks to mutually insure

\(^1\)This results is reported in several studies. It is probably most prominently emphasized in Baelen, Ferrando, Hördahl, Krylova, and Monnet (2004). For a comprehensive institutional analysis of the various segments of the Euro money market see also Hartmann, Manna, and Manzanares (2001) and for a description of the most recent developments in that market see European Central Bank (2005). Barros, Berglöf, Fulghieri, Gual, Mayer, and Vives (2005) also emphasize the disappointing degree of financial integration in most other financial markets in the Euro area particularly the retail financial markets.
against negatively correlated regional liquidity shocks. However, when analyzing the structure of the interbank network both do not take the possibility of aggregate liquidity shocks into account. Thus, any network that connects–directly or indirectly–all entities of the financial system provides an efficient mechanism to share the risk of liquidity shocks. In a second step they introduce an aggregate liquidity shock that occurs with probability zero and study its impact on arbitrary interbank networks. In contrast, we analyze the decision of banks to provide each other with interbank deposits in order to share liquidity risk taking the possibility of aggregate liquidity shocks and financial contagion into account. This permits us to derive the optimal integration decision depending on (i) the underlying stochastic structure of liquidity needs (ii) individual’s risk preferences and (iii) the state of the technology.

Focusing on a two-regional case we also discuss the efficiency of different interbank arrangements in dealing with contagion and in implementing an efficient risk sharing. We show that the rules concerning the seniority of household vs. interbank debt when liquidating a bank are crucial for the efficiency of the risk sharing on the interbank market. Under certain conditions, the interbank market can only implement constraint efficient risk sharing if interbank deposits are junior to households claims and interbank debt cannot be netted. But precisely these arrangements bring about the risk of contagion between banks. Thus implementing a constraint efficient risk sharing through an interbank market is necessarily associated with the risk of financial contagion. In this respect our paper is also related to Leitner (2005) who argues that banks put themselves at risk of contagion in order to credibly commit to bail each other out.

Another main contribution of our papers is to identify further institutional improvements that are not available in a simple interbank market setup. We interpret the interbank market as an incentive mechanism, dealing with banks private information about their respective liquidity needs. We show that the optimal mechanism also enables banks to share the benefits of excessive liquidity. However, the corresponding institutional arrangements require decisions of the banks that go beyond simply deciding whether

\[ \text{In Allen and Gale (2000) there exist four regions two of which experience a positive and two of which a negative liquidity shock. Aggregate liquidity demand is constant with probability one. This specification guarantees that there is a need for cross regional liquidity sharing and it permits to study financial contagion which is the main focus of their paper.} \]
to liquidate its assets with another bank or not. Therefore, we argue that these more complex mechanisms that use a larger strategy space essentially reflect a merged bank and not what is usually labelled an interbank market.

The remainder of the paper is organized as follows: After specifying the assumptions section 3 derives the utility that separate banks provide depositors. In section 4 we derive the optimal deposit contract and the utility that an integrated financial system generates. In this section we assume that in order to share regional liquidity risks banks have to use interbank deposits that are junior to households deposits. Section 5 analyzes the trade-off between having either a partitioned or an integrated financial systems. In section 6 we analyze the equilibrium in the interbank market in detail. We show under which conditions interbank deposit have to be junior to households’ claims in order to allow for cross-regional risk sharing. Finally, we derive in section 7 the optimal mechanism for risk sharing between banks. In section 8 we argue that this mechanism mainly reflects cross-country mergers and show to what extent this mechanism leads to a preferable allocation as compared to the one provided by junior interbank deposits. Section 9 concludes and points out some issues for further research.

2 The model

2.1 Depositors

Consider an economy with three dates \((t = 0, 1, 2)\) and two symmetrical regions \(A\) and \(B\). Both regions contain a bank and a continuum of mass 1 of households each endowed with 1 unit of the single consumption good. Households are subject to a preference shock that realizes in \(t = 1\) and that is not publicly observable. Ex-ante (in \(t = 0\)) households only know that they will be impatient \((\theta_i = 1)\) with probability \(q\). In that case they can only derive utility from consumption in \(t = 1\). With probability \((1 - q)\) they know that they will turn out to be patient and only want to consume in \(t = 2\). Thus households’ ex-ante expected utility is given by the following function

\[
E[U(c_1; c_2)] = E[\theta_i u(c_1) + (1 - \theta_i) u(c_2)]
\]
with

\[ u(c_t) = \frac{1}{1 - \gamma} c_t^{1-\gamma} \quad \text{and} \quad \gamma > 1. \]

The regional fraction of impatient households is

\[ \int_0^1 \theta_i \, di = q. \]

However, this fraction of patient consumers (i.e. the probability that a given household turns out to be impatient) in each region is itself subject to shocks. It can either turn out to be 0, \( \frac{1}{2} \) or 1. The joint probability distribution of the liquidity shocks in the two regions is given by the following table. Thus the probability that all households in both regions turn out to be patient, for instance, is given by \( b \).

<table>
<thead>
<tr>
<th></th>
<th>( q_A = 0 )</th>
<th>( q_A = \frac{1}{2} )</th>
<th>( q_A = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_B = 0 )</td>
<td>( b )</td>
<td>( d )</td>
<td>( e )</td>
</tr>
<tr>
<td>( q_B = \frac{1}{2} )</td>
<td>( d )</td>
<td>( a )</td>
<td>( f )</td>
</tr>
<tr>
<td>( q_B = 1 )</td>
<td>( e )</td>
<td>( f )</td>
<td>( c )</td>
</tr>
</tbody>
</table>

### 2.2 Investment technology

There is only one direct investment technology available in the economy. This technology can be liquidated in \( t = 1 \) at no costs.\(^3\) However, it only pays a positive interest in \( t = 2 \):\(^3\)

<table>
<thead>
<tr>
<th></th>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
<th>( t = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>finished</td>
<td>-1</td>
<td>0</td>
<td>( R &gt; 1 )</td>
</tr>
<tr>
<td>liquidated</td>
<td>-1</td>
<td>+1</td>
<td>0</td>
</tr>
</tbody>
</table>

In \( t = 0 \) households can invest in the technology. Because it is not observable whether a particular household is patient or impatient, there is no direct insurance mechanism against liquidity risks available.

Besides direct investment households can deposit their endowment at a bank. Banks offer deposit contracts, that specify the promised repayment \( d_1 \) if deposits are withdrawn in \( t = 1 \). The residual of banks’ assets (after repaying outstanding interbank debt) in

\(^3\)Note that this simplifying assumption compared to the standard Diamond and Dybvig (1983)-model can be seen as a short-cut for the existence of a liquid financial market in \( t = 1 \) at which agents from other than the considered regions and banks from the region \( A \) and \( B \) trade liquidity against financial claims promising a \( t = 2 \)-return \( R \) at the arbitrage free price \( p = 1 \) (see Fecht (2004) for a detailed description).
\( t = 2 \) will be repaid to households that keep their deposits until \( t = 2 \). \( d_2 \) specifies the repayment patient depositors can expect under normal circumstances (i.e. those cases where no regional liquidity shock occurs; \( q_i = 1/2 \)) There is one bank in each region. However, banking markets are contestable. Therefore banks are forced to offer the deposit contract that maximizes the expected utility of depositors.

If a bank cannot serve all withdrawals in \( t = 1 \) all depositors—those that initially withdrew and those that did not—receive the same pro-rata repayment. Thus, we do not consider any kind of sequential service constraint. Hence, we exclude expectation driven bank runs of depositors.

### 2.3 The interbank market

We analyze under which condition it is reasonable for banks in the two different regions to provide each other with interbank deposits. Interbank deposit contracts specify a repayment \( \{d^B_1; d^B_2\} \) contingent upon whether the deposits are withdrawn in \( t = 1 \) or held until \( t = 2 \). The regional liquidity shock is private information to the banks in the respective region. Banks cannot observe the liquidity shock in other regions. Thus interbank deposit contracts cannot be contingent on regional liquidity shocks. In order to provide a means of cross-regional risk sharing interbank deposit contracts have to be incentive compatible.

We make the following general assumptions about the institutional setup of this market:

1. First, we assume that interbank debt is junior to households’ deposits in \( t = 1 \). This means that if a bank fails then all households’ deposits (those households that wanted to withdraw and those that initially wanted to keep their deposits) are served first before any repayment is made on interbank debt. Note that this requires a gross-settlement of interbank debt—i.e. interbank positions cannot be netted. \(^4\)

2. Next, we assume that if a bank is illiquid then depositors force the bank management

\(^4\)Note that this is an optimal arrangement, because this gives banks the strongest incentive in this set-up to provide the other bank with liquidity. If interbank deposits were senior to households’ deposits then banks would always draw on their interbank deposits first to provide liquidity to other banks or households irrespective of their liquidity shock. See section 6 for a proof of this argument.
to immediately (still within $t = 1$) withdraw deposits from other banks. Thus in case of illiquidity of one bank interbank deposits are withdrawn irrespective of the original decision of the bank management.

3. Finally, we assume that depositors cannot observe whether a bank withdraws interbank deposits or not. Thus the withdrawal decision of depositors in one region cannot depend on the decision to draw on the interbank deposits of either bank.

Market contestability ensures that banks choose interbank market arrangements that maximize depositors’ utility.

2.4 Preferences of bank managers

We make the following behavioral assumption about bank managers. Given a consumer deposit contract $D$ and given an interbank market contract $D^B$ bank managers choose a state contingent plan for their behavior on the interbank market that maximizes expected utility of their non-bank customers.

This assumption needs some further justification because, at the stage where bank managers observe their respective liquidity shock, they cannot contractually be forced to act in a certain state dependent way (i.e. withdraw deposits only in certain states). However, contractual arrangements that align bankers’ and depositors’ interests can easily be implemented. Consider e.g. a contract that guarantees the manager a transfer which is an increasing function $f(\cdot)$ of the sum actually paid out to each individual depositor in $t = 1$ and $t = 2$. In particular let

$$f(\cdot) = \varepsilon \cdot \frac{1}{1 - \gamma} d^{1-\gamma},$$

where $d$ denote the sum paid out and $\varepsilon$ is arbitrarily small. This contract obviously aligns the interests of the banker and those of the depositors from the ex-ante point of view.
2.5 Equilibrium concept

An interbank market with a given contract $D^B$ induces a Bayesian game among bank managers. In this game a strategy is a state contingent plan for withdrawal decisions

$$w(q) : \{0, 1/2, 1\} \rightarrow \{0, 1\}.$$ 

An equilibrium is a combination of deposit contracts and bank strategies that are mutually consistent and compatible with competitive banking behavior.

**Definition 1** An equilibrium consists of

(i) A deposit contract $D = (d_1, d_2)$,

(ii) An interbank deposit contract $D^B = (d^B_1, d^B_2)$ and

(iii) bank strategies $w(q) : \{0, 1/2, 1\} \rightarrow \{0, 1\}$,

such that

(i) Bank strategies are a Bayesian equilibrium of the interbank market game given the interbank contract $D^B$.

(ii) The deposit contract $D$ maximizes consumer utility given $D^B$ and the withdrawal behavior of all banks.

2.6 Institutional choice

Whether or not financial institutions engage on the interbank market will depend on whether the equilibrium outcome yields higher utility to consumers than under financial separation.

3 Equilibrium with separated banks

Because of the contestability of the banking market a separated bank will offer the deposit contract $(d_1^*, d_2^*)$ that maximizes the expected utility subject to the per capita budget constraint. Given $d_1^S > 1$ the bank is always illiquid if $q = 1$. In that case the bank can only repay 1 to each depositor. If $q = 0$ the bank can finish all its projects and
pay $R$ in $t = 2$ to all depositors. Thus the contractual repayment is only important if some households want to withdraw while others want to keep their deposits until $t = 2$. Only for $q = \frac{1}{2}$ the bank actually has to pay the contracted amount $d_1$ to its impatient depositors. It immediately follows from the per capita budget constraint that the patient depositors receive a repayment $2R - Rd_1$ in $t = 2$. Thus the optimal deposit contract that the bank can offer maximizes the expected utility function

$$E[U^S(d_1^S)] = (a+d+f) \left[ \frac{1}{2} u(d_1^S) + \frac{1}{2} u(2R - Rd_1^S) \right] + (b+d+e)u(R) + (c+e+f)u(1).$$

Since $u'(c) = c^{-\gamma}$ it is easy to see that the optimal deposit contract solves

$$\frac{1}{2} (d_1^S)^{-\gamma} = \frac{R}{2} (2R - Rd_1^S)^{-\gamma},$$

and is given by

$$d_1^* = \frac{2}{R^{(1-\gamma)/\gamma} + 1}. \quad (2)$$

### 4 Equilibrium with an interbank market

The interbank market allows banks to provide each other with interbank deposits that specify the provision of liquidity at date 1. In this section we consider a potential equilibrium where (i) deposit contracts are given by $D = (d_1^M, d_2^M)$ and (ii) interbank contracts specify that each bank has the right to either withdraw $d_1^M/2$ at date 1 or $d_2^M/2$ at date 2 from the other bank.$^5$ Since banks’ liquidity needs are not observable for other players, the contract leads to a Bayesian game among banks. In what follows we will consider the strategy profile where each bank withdraws its deposits at date 1 if and only if the early liquidity shock has realized. In section 6 we prove that this is indeed a Bayesian Nash equilibrium of the interbank market game.

Depositors’ equilibrium payoffs can be characterized as follows: When both regions are not hit by any shock, depositors realize their desired consumption levels $d_1^M$ at date 1 or $d_2^M$ at date 2. When both regions experience a late shock, consumers get $R$ at date

\[5\] Other potential equilibria (with different levels of interbank deposits) can be ruled out since they would imply that withdrawals on the interbank market are either (i) not sufficient to finance additional liquidity needs or (ii) too large so that a withdrawal even leads to the liquidation of a bank with excess liquidity.
2. When shocks in different directions occur the levels $d_1^M$ or $d_2^M$ are realized. A liquidity crises occurs in case of a liquidity shortage in both regions. Moreover, if in addition $d_1^M \geq 4/3$ both banks end up being illiquid in those cases where only one region is hit by a negative liquidity shock. Finally, no cross-regional risk-sharing is possible in cases in which one region has no impatient depositors while a fraction of 1/2 is impatient in the other region.

Thus, households’ expected utility in an integrated financial systems is given by

$$E \left[ U^M (d_1^M) \right] = (a + d + 2e) \left[ \frac{1}{2} u (d_1^M) + \frac{1}{2} u (2R - Rd_1^M) \right]$$

$$+ (b + d) u (R) + (c + 2f) u (1).$$

It is easy to see that the deposit contract that maximizes households’ expected utility in a financial system that only implements cross-regional risk sharing over the interbank market is the same as in a financial system with separated banks

$$d_1^M = \frac{2}{R^{(1-\gamma)/\gamma} + 1}.$$  

Note, however, that this is only the optimal deposit contract if $d_1^M > \frac{4}{9}$ or put differently if

$$R > 2^{\frac{2}{2-\gamma}}. \tag{3}$$

Only for these parameter settings contagion will occur in the integrated financial system. If (3) does not hold then an optimally integrated financial system also provides a means for risk-sharing for those cases in which one region is hit by a negative liquidity shock while in the other region half of the households turn out to be patient. Consequently, under these conditions linking the two regions does not bring about any costs. However, if (3) holds then the decision to integrate the two regional financial systems is non-trivial. This case is discussed in the following section.

5 Optimal financial integration

Obviously, it is efficient to integrate the two regions in one financial systems if for the optimal deposit contract $d_1^*$ given by (2)

$$E \left[ U^M (d_1^*) \right] > E \left[ U^S (d_1^*) \right]. \tag{4}$$
Defining

\[ E[U^G(d_1^*)] = \frac{1}{2(1-\gamma)} \left( \frac{2}{R^{(1-\gamma)/\gamma} + 1} \right)^{1-\gamma} + \left( \frac{2R^{1/\gamma}}{R^{(1-\gamma)/\gamma} + 1} \right)^{1-\gamma}, \quad (5) \]

it is easy to see that (4) requires that

\[ e \left( 2E[U^G(d_1^*)] - u(R) - u(1) \right) > f \left( E[U^G(d_1^*)] - u(1) \right). \quad (6) \]

The LHS of (6) are the benefits from diversification and the RHS are the costs from contagion. Thus if the benefits overcompensate the costs an integration of the two regions in one financial system is preferable.

Obviously, (6) can be transformed in

\[ e \left( u(R) - E[U^G(d_1^*)] \right) < (e - f) \left( E[U^G(d_1^*)] - u(1) \right). \]

We have:

**Proposition 2** A separated financial system is always preferable if \( e \leq f \). For \( e > f \) an integrated financial system for the two regions is preferable if

\[ (2e - f) \left( \frac{R^{(1-\gamma)/\gamma} + 1}{2} \right)^{\gamma} > (e - f) + eR^{1-\gamma}. \quad (7) \]

**Proof.** Reinserting (5) in (6) and rearranging yields (7).

More details about the optimality of different arrangements can be derived from the following example.

**Example 3** For \( \gamma = 2 \) inequality (7) can be reduced to

\[ (4e - 2f) R^{-1/2} > (2e - 3f) + (2e + f) R^{-1}. \]

This holds for all

\[ R \in \left[ 1, \frac{(2e + f)^2}{(2e - 3f)^2} \right]. \]

Thus taking (3) into account an integrated financial system – even though it brings about a risk of financial contagion – is preferable, if

\[ R \in \left[ 4, \frac{(2e + f)^2}{(2e - 3f)^2} \right], \]

and this is a non-empty set. In contrast, for

\[ R > \frac{(2e + f)^2}{(2e - 3f)^2} \]

separation is preferable if \( e > f \).
Accordingly an increase in the long run rate of return $R$ makes financial separation more desirable. A higher rate of return $R$ raises the cost of financial contagion because contagion always reduced all consumer’s payoffs in the regions concerned to 1.

6 Interbank market and the implementation of constraint efficient risk-sharing

6.1 Payoffs of the interbank market game

In this section we analyze in detail the equilibrium behavior of banks on the interbank market. Given the institutional arrangement of the interbank market the state contingent pay-offs of both banks given their decisions to either withdraw or keep their deposits with the other bank are summarized in table 1 and 2.

Table 1: State contingent repayments to depositors given banks’ withdrawal decision

<table>
<thead>
<tr>
<th>${q_A; q_B}$</th>
<th>Prob</th>
<th>${s_A; s_B} = {1; 1} \phantom{1}^1$</th>
<th>${s_A; s_B} = {0; 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0; 0}$</td>
<td>$b$</td>
<td>${0; \frac{1}{2} (d_1 + d_2)}; {0; \frac{1}{2} (d_1 + d_2)}$</td>
<td>${0; d_2}, {0; R - \frac{1}{2} (d_2 - d_1)}$</td>
</tr>
<tr>
<td>${0; \frac{1}{2}}$</td>
<td>$d$</td>
<td>${0; \frac{1}{2} d_2 + \varpi}; {d_1; d_1}$</td>
<td>${0; d_2}, {d_1; R - \frac{1}{2} d_2}$</td>
</tr>
<tr>
<td>${0; 1}$</td>
<td>$e$</td>
<td>${0; \frac{1}{2} d_2 + \varpi}; {d_1; 0}$</td>
<td>${0; d_2}, {d_1; 0}$</td>
</tr>
<tr>
<td>${\frac{1}{2}; 0}$</td>
<td>$d$</td>
<td>${d_1; d_1}; {0; \frac{1}{2} d_2 + \varpi}$</td>
<td>${d_1; d_1}, {0; \frac{1}{2} d_2 + \varpi}$</td>
</tr>
<tr>
<td>${\frac{1}{2}; \frac{1}{2}}$</td>
<td>$a$</td>
<td>${1; 1}; {1; 1}$</td>
<td>${1; 1}; {1; 1}$</td>
</tr>
<tr>
<td>${\frac{1}{2}; 1}$</td>
<td>$f$</td>
<td>${1; 1}; {1; 0}$</td>
<td>${1; 1}; {1; 0}$</td>
</tr>
<tr>
<td>${1; 0}$</td>
<td>$e$</td>
<td>${d_1; 0}; {0; \frac{1}{2} d_2 + \varpi}$</td>
<td>${d_1; 0}; {0; \frac{1}{2} d_2 + \varpi}$</td>
</tr>
<tr>
<td>${1; \frac{1}{2}}$</td>
<td>$f$</td>
<td>${1; 0}; {1; 1}$</td>
<td>${1; 0}; {1; 1}$</td>
</tr>
<tr>
<td>${1; 1}$</td>
<td>$c$</td>
<td>${1; 0}; {1; 0}$</td>
<td>${1; 0}; {1; 0} \phantom{5})$</td>
</tr>
</tbody>
</table>

$^1$) $s_i = 1 : \text{Bank } i \text{ withdraws}$

Because of the assumed gross settlement of interbank liabilities, both banks have to liquidate some of their assets in order to repay their interbank deposits even if they withdraw their deposits in the same period. Assume that both banks promise a repayment $\{d_1^{IB}; d_2^{IB}\} = \{\frac{1}{2} d_1^*; \frac{1}{2} d_2^*\}$ on interbank deposits. Thus in order to repay the deposits of bank $B$ if bank $B$ withdraws in $t = 1$ bank $A$ has to liquidate $\frac{1}{2} d_1$ of its assets. At the same time...
time bank B has to liquidate \( \frac{1}{2}d_1 \) of its deposits if bank A also withdraws in \( t = 1 \). If both banks are hit by a positive liquidity shock (\( \{q_A; q_B\} = \{0; 0\} \)) they are both able liquidate this amount without collapsing. Thus given that both banks withdraw (\( \{s_A; s_B\} = \{1; 1\} \)) in this state the repayment \( \{d_1; d_2\} \) to depositors of bank A and bank B, respectively, is given by \( \{0; \frac{1}{2}(d_1 + d_2)\} \).

In contrast, in those states where \( \{q_A; q_B\} = \{0; \frac{1}{2}\} \) only bank A can fulfill its payment obligation to bank B in \( t = 1 \). Because \( d_1 > 1 \) bank B is illiquid and has to serve households deposits first. After receiving the payment from bank A the overall liquidity available to bank B is given by: \( 1 + \frac{1}{2}d_1 \). Thus the fractional repayment \( \varpi \) that bank A will receive from bank B follows from \( d_1 + \varpi = 1 + \frac{1}{2}d_1 \) and is (for the optimal deposit contract with \( d_1 = \frac{2R}{R+R\gamma} \) ) given by \( \varpi = \frac{R\gamma}{R+R\gamma} < \frac{1}{2}d_1 \). Thus bank A can only repay \( R(1 - \frac{1}{2}d_1) + \varpi = \frac{1}{2}d_2 + \varpi \) to its depositors.\(^6\)

Similarly, if both banks withdraw in states with \( \{q_A; q_B\} = \{0; 1\} \) bank A has to liquidate \( \frac{1}{2}d_1 \) in order to fulfill its payment obligation. Bank B is illiquid has to serve households deposits first. Just like in the above described case only the remaining liquidity \( \varpi \) can be paid to bank A.

If neither bank faces a liquidity shock (\( \{q_A; q_B\} = \{\frac{1}{2}; \frac{1}{2}\} \)) and both banks withdraw their interbank deposits then both banks fail. They cannot raise sufficient funds in \( t = 1 \) by liquidating their projects to repay the impatient depositors and honor the interbank deposits. In that case all projects are liquidated and households deposits are served pro-rata – i.e. each depositor receives a repayment of 1.

The same happens if both banks withdraw their interbank deposits in all those states in which at least one bank faces a negative liquidity shock. Both banks turn out to be illiquid and can only repay a pro-rata repayment of 1 to households. However, in those cases both banks will always fail irrespective of their decision to withdraw interbank deposits or not.

For instance, even if in the state \( \{q_A; q_B\} = \{1; \frac{1}{2}\} \) bank A would not decide to withdraw it would turn out to be illiquid because of the withdrawals of households. Depositors will

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\(^6\)Note that this follows from the fact that both banks can only store the repayment on interbank deposits until \( t = 2 \) when the funds are needed to repay the depositors and therefore \( R(1 - \frac{1}{2}d_1) + \frac{1}{2}d_1 = \frac{1}{2}(d_1 + d_2) \)

\(^7\)Note that if households could observe the interbank payments and the regional liquidity shock depositors in region A would run if \( d_1 > \frac{1}{2}d_2 + \varpi \). Thus depositors in region A would end up with \( \{1; 1\} \) just like depositors in region B in that case.
therefore (still within \( t = 1 \)) force bank A to withdraw its interbank deposits from bank B in order to raise some additional liquidity. As soon as bank A withdraws its interbank deposits bank B will also turn out to be illiquid. Thus in those states both banks will be forced to withdraw irrespective of their initial decision. Consequently, both banks will finally fail and repay only the pro-rata repayment 1 to each depositor.

It is easy to see that in all those states with \( q_A = \frac{1}{2} \) and \( q_A = 1 \) bank A will always be illiquid if bank B withdraws. Therefore, its own decision to withdraw or not does not matter for the repayment to its depositors.

In contrast, if both banks are hit by a positive liquidity shock (\( q_A = q_B = 0 \)) and bank B withdraws while bank A keeps its interbank deposits bank A has to liquidate \( \frac{1}{2} d_1 \) of its investment to repay bank B. But because bank B is solvent bank A receives \( \frac{1}{2} d_2 \) in \( t = 2 \) from bank B. Thus bank A can repay \( R(1 - \frac{1}{2} d_1) + \frac{1}{2} d_2 = d_2 \). In contrast, bank B receives \( \frac{1}{2} d_1 \) and can only store this liquidity. It does not have to liquidate any investments but has to pay \( \frac{1}{2} d_2 \) to bank A in \( t = 2 \). Thus bank B can repay \( R - \frac{1}{2} d_2 + \frac{1}{2} d_1 \). In general, it is easy to see that if bank A has a liquidity surplus and does not withdraw its interbank deposits while bank B withdraws bank B will always be liquid and able to repay \( \frac{1}{2} d_2 \) in \( t = 2 \) to bank A.

Table 2: State contingent repayments to depositors given banks’ withdrawal decision (cont.)

<table>
<thead>
<tr>
<th>{q_A; q_B}</th>
<th>Prob</th>
<th>{s_A; s_B} = {1; 0}</th>
<th>{s_A; s_B} = {0; 0}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0; 0}</td>
<td>b</td>
<td>{0; R - \frac{1}{2} (d_2 - d_1)}, {0; d_2}</td>
<td>{0; R}; {0; R}</td>
</tr>
<tr>
<td>{0; \frac{1}{2}}</td>
<td>d</td>
<td>{0; \frac{1}{2} d_2 + \omega}, {d_1; d_1}</td>
<td>{0; R}; {d_1; d_2}</td>
</tr>
<tr>
<td>{0; 1}</td>
<td>e</td>
<td>{0; \frac{1}{2} d_2 + \omega}; {d_1; 0}</td>
<td>{0; \frac{1}{2} d_2 + \omega}; {d_1; 0}</td>
</tr>
<tr>
<td>{\frac{1}{2}; 0}</td>
<td>d</td>
<td>{d_1; R - \frac{1}{2} d_2}, {0; d_2}</td>
<td>{d_1; d_2}; {0; R}</td>
</tr>
<tr>
<td>{\frac{1}{2}; \frac{1}{2}}</td>
<td>a</td>
<td>{1; 1}; {1; 1}</td>
<td>{d_1; d_2}; {d_1; d_2}</td>
</tr>
<tr>
<td>{\frac{1}{2}; 1}</td>
<td>f</td>
<td>{1; 1}; {1; 0}</td>
<td>{1; 1}; {1; 0}</td>
</tr>
<tr>
<td>{1; 0}</td>
<td>e</td>
<td>{d_1; 0}, {0; d_2}</td>
<td>{d_1; 0}; {0; \frac{1}{2} d_2 + \omega}</td>
</tr>
<tr>
<td>{1; \frac{1}{2}}</td>
<td>f</td>
<td>{1; 0}; {1; 1}</td>
<td>{1; 0}; {1; 1}</td>
</tr>
<tr>
<td>{1; 1}</td>
<td>c</td>
<td>{1; 0}; {1; 0}</td>
<td>{1; 0}; {1; 0}</td>
</tr>
</tbody>
</table>

If both banks do not withdraw their interbank deposits in \( t = 1 \) (\( \{s_A; s_B\} = \{0; 0\} \))
then no funds are transferred between the two banks if both stay liquid. If either bank fails to repay its depositors then depositors can force the bank to collect its interbank deposits. This happens, for instance, if \( \{q_A; q_B\} = \{0; 1\} \). Because bank B is illiquid it will be forced to withdraw its interbank deposits. In the liquidation of bank B bank A can only recoup a repayment of \( \varpi \) on its interbank deposits.

Finally, it is important to note that bank A can only survive \( \{q_A; q_B\} = \{1/2; 1\} \) if neither bank A nor bank B withdraws.

### 6.2 Equilibrium

Due to the fact that liquidity needs are not observable among banks, the interbank market induces a Bayesian game among banks at \( t=1 \). A bank’s strategy maps the realized liquidity demand (the fraction of early withdrawing depositors) at \( t=1 \) into a decision to withdraw or to keep the interbank deposit.

Our first main result is that indeed an interbank market with the above institutional features may allow for a limited extend of risk sharing. This means that negative liquidity shocks can be compensated by positive shocks. However, excess liquidity is not equally distributed among regions.

**Proposition 4** The interbank market game has a Bayesian Nash equilibrium where both banks withdraw at \( t=1 \) if and only if they experience an early liquidity shock. In this equilibrium the interbank market (i) provides an optimal risk sharing for states with opposed regional liquidity shocks, (ii) does not implement a cross regional risk sharing for states with excess aggregate liquidity, (iii) causes contagion in states with aggregate liquidity shortages.

**Proof.** Given that bank B only withdraws its interbank deposits if it is hit by a negative liquidity shock \( (q_B = 1) \) we have to show that it is optimal for bank A to withdraw its interbank deposits in \( t=1 \) if and only if the bank is hit by a negative liquidity shock \( (q_A = 1) \). From table 1 and 2 it is easy to derive the following incentive compatibility constraints and verify under which conditions they hold.

First consider \( q_A = 0 \). Expecting that bank B only withdraws its interbank deposits...
if \( q_B = 1 \) bank \( A \) keeps its interbank deposits until \( t = 2 \) iff

\[
(b + d) \, U(0; R) + e \, U(0; d_2) \geq \]

\[
b \, U\left(0; R - \frac{1}{2} (d_2 - d_1)\right) + (d + c) \, U\left(0; \frac{1}{2} d_2 + \varpi\right).
\]

(IC1) always hold because \( R > R - \frac{1}{2} (d_2 - d_1) \) and \( R > d_2 > \frac{1}{2} d_2 + \varpi \).

For \( q_A = \frac{1}{2} \) bank \( A \) will not withdraw its interbank deposits in \( t = 1 \) iff

\[
(d + a) \, U(d_1; d_2) + f \, U(1; 1) \geq \]

\[
d \, U\left(d_1; R - \frac{1}{2} d_2\right) + (a + f) \, U(1; 1).
\]

Since \( d_2 > d_1 > 1 \) (IC2) always holds for the optimal deposit contract that banks offer to households\(^8\) if \( R < 2^{\gamma/(\gamma - 1)} \). Even if \( R > 2^{\gamma/(\gamma - 1)} \) and therefore \( U(d_1; R - \frac{1}{2} d_2) > U(d_1; d_2) \) (IC2) always holds if \( a \) is sufficiently large relative to \( d \):

\[
a \, [U(d_1; d_2) - U(1; 1)] \geq d \left[ U\left(d_1; R - \frac{1}{2} d_2\right) - U\left(d_1; d_2\right)\right]
\]

Finally consider \( q_A = 1 \). Given that bank \( B \) only withdraws if \( q_B = 1 \) the repayment bank \( A \)'s depositors receive is independent of the decision of bank \( A \) to withdraw its interbank deposits or not because if the bank does not withdraw its interbank deposits the bank fails and will be forced by the depositors to withdraw. Thus the incentive compatibility constraint is given by

\[
e U(d_1; 0) + (f + c) \, U(1; 0) \geq e U(d_1; 0) + (f + c) \, U(1; 0),
\]

and holds with an equality. \( \blacksquare \)

6.3 Seniority rules

We are now able to study the role of seniority rules in interbank arrangements.

**Proposition 5** For sufficiently large \( R \) and sufficiently small \( b \) cross-regional risk sharing cannot be implemented with interbank deposits that are senior to households claims.

\(^8\)Remember that the optimal deposit contract that banks offer to households always implies a risk sharing \( \frac{d^2}{d_1} = R^{1/\gamma} \) for integrated as well as separated financial systems given that banks cannot implement a cross regional risk sharing in cases of positive aggregate liquidity shocks.
Proof. If interbank deposits were senior bank $A$’s incentive compatibility constraint to keep deposits until $t = 2$ if $q_A = \frac{1}{2}$ would change from (IC2) to (IC2')

$$(d + a) U(d_1; d_2) + fU(1; 1) \geq dU\left(d_1; R - \frac{1}{2}d_2\right) + (a + f) U(d_1; d_2), \quad \text{(IC2')}$$

which does not hold for $R > 2^{\gamma/(\gamma-1)}$ because

$$d \left[U\left(d_1; R - \frac{1}{2}d_2\right) - U(d_1; d_2)\right] > f \left[U(d_1; d_2) - U(1; 1)\right].$$

, which follows from the fact that the LHS is positive while the RHS is negative.

If interbank deposits were senior both banks would have an incentive to withdraw their deposits irrespective whether or not they are actually in need for liquidity--i.e. independent of their respective $q_i$.

Taking into account that it is preferable for both banks to withdraw their interbank deposits for $q_i = \frac{1}{2}$ the incentive compatibility constraint of bank $A$ to keep their interbank deposits until $t = 2$ if $q_A = 0$ changes to (IC1')

$$bU(0; R) + (d + e) U(0; d_2) \geq bU\left(0; R - \frac{1}{2}(d_2 - d_1)\right) + (d + e) U(0; R) \quad \text{(IC1')}$$

Obviously, for $(d + e)$ being sufficiently high relative to $b$ it is also for banks that are hit by a positive liquidity shock preferable to withdraw their interbank deposits in $t = 1$.

In contrast, if (IC2') does not hold but (IC1') then an interbank market with senior interbank deposits can implement some cross regional risk sharing. In that case the interbank market would allow for cross-regional risk sharing without bringing about the risk of contagion. Because if interbank deposits are senior and banks only keep their deposits until $t = 2$ if they are hit by a positive liquidity shock then there are no spillovers of the illiquidity of one bank to the other in those cases with $\{q_A; q_B\} = \{\frac{1}{2}; 1\}$ and $\{1; \frac{1}{2}\}$.

7 Alternative market mechanisms

So far, we took the organization of the interbank market as exogenously given. In this section we study whether there is an alternative interbank market mechanism that yields a superior outcome. We start out by assuming that consumer deposit contracts of the sort derived in section 4 already exist.
In our setup the revelation principle holds. Hence, we may restrict our analysis to direct revelation mechanisms. A direct interbank market mechanism asks both banks for a report on their realized liquidity parameter and maps the tuple of reports \((\hat{q}_A, \hat{q}_B)\) into a decision \(x\), i.e.

\[ x = f (\hat{q}_A, \hat{q}_B). \]

The decision \(x\) consists of four elements:

- A transfer \(t_1\) from bank 1 to bank 2 in period 1.
- A transfer \(t_2\) from bank 1 to bank 2 in period 2.
- A decision on the closure of each bank in period 1.

The interbank market studied so far does not provide an efficient risk sharing in case of a positive liquidity shock in one region and no liquidity shock \((q_i = \frac{1}{2})\) in the other region. Is it possible to improve the efficiency of the risk sharing implemented by the interbank market mechanism? Consider some alternative mechanism that has the feature that (i) no bank closes when announced aggregate liquidity would be sufficient to honor the withdrawals in period 1 (ii) with opposing liquidity needs there is a sufficient transfer to the bank with high liquidity needs in period 1 (iii) provides efficient risk sharing in cases of a positive liquidity shock in one region and no liquidity shock in the other. Such a mechanism would have to fix contingent transfers as described in table 3.

Table 3: Transfers implementing unconstraint efficient risk sharing

<table>
<thead>
<tr>
<th>((\hat{q}_A, \hat{q}_B))</th>
<th>0</th>
<th>(\frac{1}{2})</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(t_1 = t_2 = 0)</td>
<td>(0 &lt; t_1 &lt; \frac{1}{2}d_1^*)</td>
<td>(t_1 = \frac{1}{2}d_1^*)</td>
</tr>
<tr>
<td></td>
<td>(0 &gt; t_2 &gt; -\frac{1}{2}d_2^*)</td>
<td>(t_2 = -\frac{1}{2}d_2^*)</td>
<td></td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>(0 &gt; t_1 &gt; -\frac{1}{2}d_1^*)</td>
<td>(t_1 = t_2 = 0)</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>(0 &lt; t_2 &lt; \frac{1}{2}d_2^*)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>1</td>
<td>(t_1 = -\frac{1}{2}d_1^*)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>(t_2 = \frac{1}{2}d_2^*)</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
It is useful to first highlight the fundamental role of bankruptcy in providing incentives to managers.

**Lemma 6** Suppose that a bank’s date 2 value is constant. In order to ensure incentive compatibility the probability of a bank closure has to increase weakly in the announced liquidity need for period 1.

**Proof.** This follows immediately from the specification of the bank manager’s preferences.

The following proposition derives necessary properties of a social choice function $f(\cdot)$ that guarantees an efficient risk sharing in case of a positive liquidity shock in one region and no shock in the other. It shows that the social choice function has to implement a closure of all banks in those cases where the reported liquidity needs of both banks indicate an aggregate negative liquidity shock in period 1: $\hat{q}_A + \hat{q}_B \leq \frac{1}{2}$.

**Proposition 7** There is no mechanism that simultaneously satisfies the following conditions: (i) Bayesian incentive compatibility, (ii) efficient risk sharing in case of a positive liquidity shock in one region and no shock in the other (iii) the survival of one bank in case of a negative liquidity shock in one region and no shock in the other region.

**Proof.** Consider without loss of generality bank $A$. In order to implement the efficient risk sharing bank $A$ would have to pay a transfer $\frac{1}{2}d_1^*$ in states with $(\hat{q}_A, \hat{q}_B) = (0, 1)$. However, to make self-revelation incentive compatible for a bank with zero liquidity needs $(q_A = 0)$ one has to implement a transfer larger than $\frac{1}{2}d_1^*$ in case that announcements are $(\hat{q}_A, \hat{q}_B) = (\frac{1}{2}, 1)$. This implies the closure of a bank with $q_A = \frac{1}{2}$ for $(\hat{q}_A, \hat{q}_B) = (\frac{1}{2}, 1)$.

On the other hand, to avoid that bank $A$ reports a high liquidity need $(\hat{q}_A = 1)$ but actually has $q_A = \frac{1}{2}$ one has to close that bank for announced liquidity needs $(\hat{q}_1, \hat{q}_2) = (1, \frac{1}{2}).$  

Moreover, one can show that the above necessary conditions are also sufficient.

**Proposition 8** There is a mechanism that simultaneously satisfies Bayesian incentive compatibility and the efficient risk sharing in case of one positive liquidity shock. Under

---

9Note that one other option would be to make bank $A$ pay in cases of announced liquidity needs $(\hat{q}_A, \hat{q}_B) = (1, \frac{1}{2})$ and close it only for announcements $(\hat{q}_A, \hat{q}_B) = (1, 1)$. However, given the budget constraint of banks $A$ if it really has $q_A = 1$ this means to the closure bank $A$ at $(\hat{q}_A, \hat{q}_B) = (1, \frac{1}{2})$.  

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this mechanism no bank survives in case of an excessive aggregate liquidity need at date 1.

**Proof.** It is straightforward to verify that all IC constraints hold.

This shows that the interbank market mechanism from section 6 is not the optimal mechanism. Risk sharing with an aggregate positive liquidity shock can be improved. However, the mechanism described in Proposition 8 cannot be implemented by an interbank market in which only simple contracts can be traded. Simple contracts can only prespecify a certain payment $d_1^B$ on the funds withdrawn in $t = 1$ and $d_2^B$ for those kept until $t = 2$ with some other bank. Even if funds can be withdrawn fractionally such a market mechanism can never implement the optimal mechanism, i.e. it can never implement any risk sharing in cases of positive liquidity shocks.

A mechanism such as the one described in Proposition 8 would have to be more complex in situations with more than two realizations of liquidity shocks. It would actually require both banks to choose a signal from a set that is as large as the set of possible realizations of the shock and map the tuple of announcements into an outcome. Such a mechanism would establish a tight relationship among two banks that we will call a merger in what follows.

## 8 Merger

The maximum degree of risk sharing can be reached by a bank operating in both regions or by a complex mechanism such as the one described in the previous section. Such a contract can be viewed as a multi-regional bank that can use its assets to serve the withdrawal of depositors from both regions. However, given that the deposit contract promises $d_1^M > 1$ the bank is illiquid if the fraction of impatient depositors is 1 in both regions, i.e. if the aggregate fraction of early withdrawals is 1. If in addition $d_1^M > \frac{4}{3}$ the bank is also illiquid in those cases where only one region is hit by a negative liquidity shock. Thus assuming $d_1^M > \frac{4}{3}$ the bank can also only pay 1 to all depositors if the aggregate fraction of impatient depositors is $\frac{4}{3}$. Therefore, from the perspective of one region the costs of a multi-regional bank are that with probability $f$ a state occurs in which the multi-regional bank provides a channel for contagion of a negative liquidity
shock from the other region.

Obviously, similar to banks that use the interbank market described in section 6 a multi-regional bank is beneficial in those states where the regional liquidity shocks compensate each other. In these cases the bank is not illiquid because it can repay the promised amount $d^M_t$ to the aggregate fraction of $\frac{1}{2}$ of impatient depositors. Therefore, in those states that occur with probability $2e$ multi-regional banks provide a way for an inter-regional risk-sharing.

In addition, a multi-regional bank also provides a means for cross-regional risk-sharing in those states in which in one region $\frac{1}{2}$ of the depositors turn out to be impatient while in the other region all households want to consume in $t = 2$. In these states the patient depositors from the first region benefit at the expense of the patient depositors from the other region, because the bank can finish more projects and can therefore generate the higher per capita $t = 2$-repayment: $\frac{4}{3} - \frac{R}{3}d^M_t$.

In this section we ask whether the possibility of a bank merger changes the result that financial decentralization may be superior to financial integration. In general the optimal deposit contract is different under a merger than under a financial integration over the interbank market. Given $d^M_t > \frac{4}{3}$ the expected utility of depositors is given by

$$E[U^M(d^M_t)] = (a + 2e)\left[\frac{1}{2}u(d^M_t) + \frac{1}{2}u(2R - Rd^M_t)\right]$$

$$+ 2d\left[\frac{1}{4}u(d^M_t) + \frac{3}{4}u\left(\frac{4}{3}R - \frac{1}{3}Rd^M_t\right)\right]$$

$$+ bu(R) + (c + 2f)u(1) .$$

It follows from the first order conditions that the optimal deposit contract solves

$$(a + 2e + d)\frac{1}{2}(d^M_t)^{-\gamma} = dR\left(\frac{4}{3}R - \frac{1}{3}Rd^M_t\right)^{-\gamma} + (a + 2e)\frac{R}{2}(2R - Rd^M_t)^{-\gamma} .$$

(8)

Comparing (8) and (1) it is easy to see that the optimality condition for the deposit contract offered by a multi-regional bank is identical to the one for the deposit contract offered by regionally separated banks if $d = 0$. Consequently, in that case a multi-regional bank will offer the same deposit contract as regionally separated banks.\textsuperscript{10} Hence we have

\textsuperscript{10}As is shown in the appendix for $d > 0$ the deposit contract offered by a multi-regional bank will promise a higher repayment in $t = 1$. Thus for $d > 0$ it follows that $d^M_t > d^S_t$.
Proposition 9 For $d > 0$ integration over the interbank market yields lower consumer welfare than a cross-regional bank merger. However, there are risk structures $(a, b, c, d, e, f)$ such that a bank merger delivers lower utility to consumers than a separated financial system.

9 Conclusion

In this paper we have studied the trade-off between cross-regional risk sharing and financial contagion. We found that for certain parameter settings a cross-regional risk sharing using interbank deposits cannot be implemented without incurring the risk of financial contagion. Moreover, not all benefits from financial integration can be realized using only an integrated interbank market. Cross-country mergers of banks provide a more efficient cross-regional risk sharing mechanism.

In analyzing the trade-off we have shown that an integrated financial system may be more preferable the lower the rate of return on long-term investment. Thus our results suggest that it is particularly more beneficial for advanced economies with a lower marginal productivity of capital to establish a common financial system. One important point for further research is to analyze this issue in-depth within a growth model.

Another issue that we have to leave for further research is to extent the model to a multi-regional setting and analyze which properties of the cross-regional distribution of liquidity shocks might lead to an imperfectly integrated financial system and which implications this has for systemic risk. Moreover, our analysis suggests that in a multilateral interbank arrangement incentives for banks not to withdraw interbank deposits decline. This might lead to an endogenous segmentation of the interbank market, that deserves further investigation.
Appendix

Optimal deposit contract of a multi-regional bank: The optimal deposit contract solves

\[
U (d^M_1; d^M_2) = (a + 2e) \left[ \frac{1}{2} u (d^M_1) + \frac{1}{2} u \left( 2R - Rd^M_1 \right) \right] \\
+ 2d \left[ \frac{1}{4} u (d^M_1) + \frac{3}{4} u \left( \frac{4}{3} R - \frac{1}{3} Rd^M_1 \right) \right] \\
+ bu (R) + (c + 2f) u (1).
\]

We have

\[
\frac{\partial U (d^N_1; d^N_2)}{\partial d^M_1} = 0
\]

\[
\Leftrightarrow (a + 2e) \left[ \frac{1}{2} u' (d^M_1) - \frac{R}{2} u' (2R - Rd^M_1) \right] + 2d \left[ \frac{1}{4} u' (d^M_1) - \frac{R}{4} u' \left( \frac{4}{3} R - \frac{1}{3} Rd^M_1 \right) \right] = 0
\]

\[
\Leftrightarrow (a + 2e + d) \frac{1}{2} u' (d^M_1) - \frac{R}{2} u' (2R - Rd^M_1) + d \left[ \frac{1}{4} u' (d^M_1) - \frac{R}{4} u' \left( \frac{4}{3} R - \frac{1}{3} Rd^M_1 \right) \right] = 0
\]

\[
\Leftrightarrow (a + 2e + d) \frac{1}{2} u' (d^M_1) = d \frac{R}{2} u' \left( \frac{4}{3} R - \frac{1}{3} Rd^M_1 \right) + (a + 2e) \frac{R}{2} u' (2R - Rd^M_1).
\]

For \( u' (c_t) = c_t^{-\gamma} \) the optimality condition can be rewritten yielding

\[
(a + 2e + d) \frac{1}{2} (d^M_1)^{-\gamma} = d \frac{R}{2} \left( \frac{4}{3} R - \frac{1}{3} Rd^M_1 \right)^{-\gamma} + (a + 2e) \frac{R}{2} \left( 2R - Rd^M_1 \right)^{-\gamma}.
\]

**Proof that** \( d^M_1 > d^S_1 \) **for** \( d > 0 \):

\[
d^M_1 > d^S_1
\]

holds if

\[
(a + 2e + d) \frac{1}{2} (d^S_1)^{-\gamma} > d \frac{R}{2} \left( \frac{4}{3} R - \frac{1}{3} Rd^S_1 \right)^{-\gamma} + (a + 2e) \frac{R}{2} \left( 2R - Rd^S_1 \right)^{-\gamma}.
\]

Given that

\[
(a + 2e) \frac{1}{2} (d^S_1)^{-\gamma} = (a + 2e) \frac{R}{2} \left( 2R - Rd^S_1 \right)^{-\gamma}
\]

(9)
this holds if

\[
(d_1^S)^{-\gamma} > R \left( \frac{4}{3} R - \frac{1}{3} R d_1^S \right)^{-\gamma}
\]

(10)

From (9) follows that

\[
(d_1^S)^{-\gamma} = R \left( 2R - Rd_1^S \right)^{-\gamma}.
\]

Since \( x^{-\gamma} \) is a strictly decreasing function in \( x \) (10) holds if

\[
2R - Rd_1^S < \frac{4}{3} R - \frac{1}{3} Rd_1^S,
\]

which is obviously true for all \( d_1^S > 1 \).
References


