Oesterreichische Nationalbank

Guidelines on Market Risk

Volume 4

Provisions for Option Risks

Guidelines on Market Risk

Volume 1: General Market Risk of Debt Instruments
2nd revised and extended edition

Volume 2: Standardized Approach Audits

Volume 3: Evaluation of Value-at-Risk Models

Volume 4: Provisions for Option Risks

Volume 5: Stress Testing

Volume 6: Other Risks Associated with the Trading Book

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The second major amendment to the Austrian Banking Act, which entered into force on January 1, 1998, faced the Austrian credit institutions and banking supervisory authorities with an unparalleled challenge, as it entailed far-reaching statutory modifications and adjustments to comply with international standards.

The successful implementation of the adjustments clearly marks a quantum leap in the way banks engaged in substantial securities trading manage the associated risks. It also puts the spotlight on the importance of the competent staff's training and skills, which requires sizeable investments. All of this is certain to enhance professional practice and, feeding through to the interplay of market forces, will ultimately benefit all market participants.

The Oesterreichische Nationalbank, which serves both as a partner of the Austrian banking industry and an authority charged with banking supervisory tasks, has increasingly positioned itself as an agent that provides all market players with services of the highest standard, guaranteeing a level playing field.

Two volumes of the six-volume series of guidelines centering on the various facets of market risk provide information on how the Oesterreichische Nationalbank appraises value-at-risk models and on how it audits the standardized approach. The remaining four volumes discuss in depth stress testing for securities portfolios, the calculation of regulatory capital requirements to cover option risks, the general interest rate risk of debt instruments and other risks associated with the trading book, including default and settlement risk.

These publications not only serve as a risk management tool for the financial sector, but are also designed to increase transparency and to enhance the objectivity of the audit procedures. The Oesterreichische Nationalbank selected this approach with a view to reinforcing confidence in the Austrian financial market and – against the backdrop of the global liberalization trend – to boosting the market’s competitiveness and buttressing its stability.

Gertrude Tumpel-Gugerell

Vice Governor
Oesterreichische Nationalbank

Today, the financial sector is the most dynamic business sector, save perhaps the telecommunications industry. Buoyant growth in derivative financial products, both in terms of volume and of diversity and complexity, bears ample testimony to this. Given these developments, the requirement to offer optimum security for clients' investments represents a continual challenge for the financial sector.

It is the mandate of banking supervisors to ensure compliance with the provisions set up to meet this very requirement. To this end, the competent authorities must have flexible tools at their disposal to swiftly cover new financial products and new types of risks. Novel EU Directives, their amendments and the ensuing amendments to the Austrian Banking Act bear witness to the daunting pace of derivatives developments. Just when it seems that large projects, such as the limitation of market risks via the EU's capital adequacy Directives CAD I and CAD II, are about to draw to a close, regulators find themselves facing the innovations introduced by the much-discussed New Capital Accord of the Basle Committee on Banking Supervision. The latter document will not only make it necessary to adjust the regulatory capital requirements, but also require the supervisory authorities to develop a new, more comprehensive coverage of a credit institution's risk positions.

Many of the approaches and strategies for managing market risk which were incorporated in the Oesterreichische Nationalbank's Guidelines on Market Risk should – in line with the Basle Committee's standpoint – not be seen as merely confined to the trading book. Interest rate, foreign exchange and options risks also play a role in conventional banking business, albeit in a less conspicuous manner.

The revolution in finance has made it imperative for credit institutions to conform to changing supervisory standards. These guidelines should be of relevance not only to banks involved in large-scale trading, but also to institutions with less voluminous trading books. Prudence dictates that risk – including the "market risks" inherent in the bank book – be thoroughly analyzed; banks should have a vested interest in effective risk management. As the guidelines issued by the Oesterreichische Nationalbank are designed to support banks in this effort, banks should turn to them for frequent reference. Last, but not least, this series of publications, a key contribution in a highly specialized area, also testifies to the cooperation between the Austrian Federal Ministry of Finance and the Oesterreichische Nationalbank in the realm of banking supervision.

Alfred Lejsek

Director General
Federal Ministry of Finance

Preface

This guideline deals with other risks inherent in options as specified in the Option Risks Regulation and, providing numerous examples and a sample portfolio, attempts to elucidate the standardized approach used to calculate the amount of regulatory capital required to back option risks in the trading book.

Section 1 provides an overview of the legal framework, introduces the most important risks inherent in options and demonstrates how these risks can be aggregated in compliance with the Option Risks Regulation.

Section 2 contains a presentation of option pricing models and sensitivities. Part I of this section focuses on a discussion of the widely used Black-Scholes model for pricing European options as well as the analytical approximation developed by Barone-Adesi and the numerical binomial tree method used to value American options. The second part of this section describes how to compute sensitivities on the basis of numerical and analytical methods and how to determine current volatility, which is one of the most important input parameters of any option pricing model.

Section 3 presents numerous examples relating to the different types of options. First, the options are valued applying the formulas introduced in Section 2. Next the sensitivities are calculated and finally the gamma and vega risk of the individual option positions are computed based on the maturity-band approach.

In Section 4 the option positions of Section 3 are combined into a sample portfolio. The gamma and vega effects are first calculated for each risk category and then for the entire portfolio.

The authors would like to thank Gerhard Coosmann, Markus Fulmek, Gerald Krenn and Ronald Laszlo for their comments, discussions and valuable suggestions. Special thanks are due to the head of the division, Helga Mramor, who promoted the production of this series of guidelines on market risk.

Vienna in September 1999

Annemarie Gaal
Manfred Plank

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1 Legal Framework

Credit institutions which do not use an internal model to calculate the regulatory capital requirement[1] for backing the options contained in the trading book may calculate the regulatory capital with respect to the general position risk associated with options pursuant to § 22e para 2 Banking Act[2] and § 22e para 3 Banking Act in connection with the Option Risks Regulation. For provisions covering the default risk of OTC options, see § 22o Banking Act. For more detailed information on default risk please refer to volume 6 of the Guidelines on Market Risk „Other Risks Associated with the Trading Book“ (Plank, 1999). § 22e para 2 Banking Act specifies the method by which the delta risk must be accounted for, whereas § 22e para 3 Banking Act in connection with the Option Risks Regulation regulates the way in which other risks inherent in options must be calculated. The Option Risks Regulation introduces a simplified procedure to take into account other risks inherent in options within the context of the calculation of the regulatory capital of the trading book. Note that the gamma risk and the vega risk must be calculated separately for each option position, i.e. also for hedging positions. Pursuant to the provisions of the Banking Act, the banking supervisory authority must be notified of the option pricing models used for these calculations.

To calculate the overall gamma and vega risk of an option portfolio, the individual positions are grouped in so-called risk categories. The gamma and vega effects of individual positions may be offset against each other only within these individual categories. In the case of options on

- foreign currency and gold, each currency pair and gold constitute separate risk categories;

- stocks, the stocks of all markets of a country form a separate risk category. If a stock is quoted on stock exchanges in several countries, the respective main market is the reference market, which can be determined either on the basis of the volume traded or of the registered office of the company;

- bonds and interest rates, broken down by currencies of the underlying instrument, each maturity band as specified in table I in cases in which the maturity-band approach is applied and each zone as specified in table II in cases in which the duration method is applied, forms a further separate risk category[3]. Please note, however, that if the general position risk inherent in debt instruments is calculated using the maturity-band approach as specified in § 22h para 2 Banking Act, the maturity bands of table I, column 2 must be applied to underlying instruments with a nominal interest rate of 3% or higher, whereas the maturity

[1] *Referred to as “own funds requirement“ in the Austrian Banking Act.*
[2] *Bankwesengesetz (BWG).*
[3] *The Option Risks Regulation will be modified accordingly.*

bands of table I, column 3 must be applied if the nominal interest rate is below 3%[4]. If an underlying has more than one maturity (e.g. in the case of swaptions, caplets and floorlets), the applicable maturity is always the longer of the two maturities, also including any relevant run-up periods.

Zones	Maturity bands		Weighting (in %)	Assumed interest-rate change (in %)	Coding
	Nominal interest rate of 3% or higher	Nominal interest rate below 3%			
Column (1)	Column (2)	Column (3)	Column (4)	Column (5)	Column (6)
Zone (1)	up to 1 month	up to 1 month	0.00	--	1
	over 1 to 3 months	over 1 to 3 months	0.20	1.00	2
	over 3 to 6 months	over 3 to 6 months	0.40	1.00	3
	over 6 to 12 months	over 6 to 12 months	0.70	1.00	4
Zone (2)	over 1 to 2 years	over 1 to 1.9 years	1.25	0.90	5
	over 2 to 3 years	over 1.9 to 2.8 years	1.75	0.80	6
	over 3 to 4 years	over 2.8 to 3.6 years	2.25	0.75	7
Zone (3)	over 4 to 5 years	over 3.6 to 4.3 years	2.75	0.75	8
	over 5 to 7 years	over 4.3 to 5.7 years	3.25	0.70	9
	over 7 to 10 years	over 5.7 to 7.3 years	3.75	0.65	10
	over 10 to 15 years	over 7.3 to 9.3 years	4.50	0.60	11
	over 15 to 20 years	over 9.3 to 10.6 years	5.25	0.60	12
	over 20 years	over 10.6 to 12.0 years	6.00	0.60	13
		over 12.0 to 20.0 years	8.00	0.60	14
		over 20 years	12.50	0.60	15

Table I: Maturity-band approach (§ 22h para 2 Banking Act)

[4] *The Option Risks Regulation will be modified accordingly.*

Zone	Modified duration	Assumed interest-rate change (in %)
1	0 to 1.0	1.0
2	over 1.0 to 3.6	0.85
3	over 3.6	0.7

Table II: Duration method (§ 22h para 3 Banking Act)

Pursuant to the provisions of the Banking Act, the delta, gamma and vega risks of options must be backed by regulatory capital.

1.1 The Delta Risk

The delta δ of an option indicates the change of the option price in the event of a small movement in the price of the underlying. In mathematical terms, the delta is the first partial derivative of the option price function with respect to the underlying asset. To calculate the delta risk of an option with respect to the underlying, the option is treated similarly to a position whose value corresponds to the value of the delta-weighted underlying. Pursuant to §s 22a through 22o Banking Act, the delta-weighted underlying must be backed by regulatory capital, after any derivative underlyings have, in a first step and in accordance with § 22e Banking Act, been decomposed into their components. For more detailed information on the decomposition of interest-rate instruments, please refer to volume 1 of the Guidelines on Market Risk "General Market Risk of Debt Instruments", 2nd revised and extended edition (Coosmann and Laszlo,1999).

1.2 The Gamma Risk

1.2.1 The Gamma Effect of an Option

The gamma γ of an option indicates the relative change of the option's delta in the event of a minor price fluctuation in the underlying asset. Mathematically, the gamma is the second partial derivative of the option price function with respect to the underlying asset. To calculate the gamma risk of an option, we have to compute the so-called gamma effect, which results from the Taylor series expansion of the option price function:

$$Gamma\ \ effect = \frac{1}{2} \cdot volume \cdot \gamma \cdot (\Delta B)^2 \tag{1.1}$$

ΔB stands for the assumed price fluctuation of the underlying, whereas the volume, as shown below, is specified in accordance with the individual categories.

The Option Risks Regulation distinguishes between four classes:

- options on stocks,
- options on foreign currency and gold,
- options on interest rates,
- options on bonds.

Specifications regarding the volume and the price fluctuation of the underlying ΔB are indicated in the table below; the applicable weighting factor is 0.04 for closely correlated currencies and 0.08 in the case of currencies which are not closely correlated.

	Stocks	**Foreign currency**	**Interest rates**	**Bonds**
Volume	Items	Face value	Face value	Face value/100
ΔB Maturity-band approach	Market value x 0.08	Market value x 0.08 or 0.04	Interest-rate change from column 5 of table I	Weighting from column 4 of table I x forward price of the bond[5]
ΔB Duration method	Market value x 0.08	Market value x 0.08 or 0.04	Interest-rate change from column 3 of table II	Duration x interest-rate change from column 3 of table II x forward price of the bond

Table III: Specifications for volume and change in the underlying

The assumptions made in tables I, II and III with respect to the change in the underlying are based on unpublished statistical surveys of the Basle Committee on Banking Supervision. Table III must be used for calculating the regulatory capital requirement in accordance with the maturity-band approach pursuant to § 22h Banking Act.

[5] *The forward price of the bond is the value of the bond at the option's exercise date as calculated today.*

1.2.2 Aggregation of the Gamma Effects

For calculating the regulatory capital required to back the gamma risk of an option portfolio, we must, in a first step, add up the individual gamma effects within a risk category in order to attain either a positive or a negative net gamma effect for each risk category. The absolute value of the total of all negative net gamma effects obtained then corresponds to the regulatory capital for backing the gamma risk. Positive net gamma effects are disregarded.

1.3 The Vega Risk

1.3.1 The Vega Effect of an Option

The vega Λ of an option indicates the change of the option price in the event of a small movement in the volatility of the underlying. In mathematical terms, the vega is the first partial derivative of the option price function with respect to volatility. For computing the vega risk of an option we have to calculate the so-called vega effect, which results from a Taylor series expansion of the option price function:

$$Vega\ \ effect = volume \cdot \Lambda \cdot \frac{volatility}{4} \tag{1.2}$$

The change of the current volatility - which must be indicated as a decimal - is assumed to be one fourth of the current volatility[6].

1.3.2 Aggregation of the Vega Effects

For calculating the regulatory capital required to back the vega risk of an option portfolio, we must, in a first step, add the individual vega effects within the individual risk categories in order to attain either a positive or a negative net gamma effect for each risk category. The total of absolute values of the net vega effects obtained then corresponds to the regulatory capital for backing the vega risk.

[6] *It must be pointed out in this context that some software systems calculate the vega based on a volatility change by one percentage point. In this case the vega must be multiplied by a factor of 100 before the formula (1.2) can be applied.*

1.4 Annex: The Taylor Series Expansion

The formulas in the Option Risks Regulation are mathematically based on the fact that under specific circumstances the value of a function which depends on several variables $x_1,\ldots,x_n$ may well be approximated in a small neighborhood of $x_1,\ldots,x_n$ by means of a polynomial function. The coefficients of this polynomial function are given by the partial derivatives of the function at $x_1,\ldots,x_n$. Formally, this can be expressed in the following way:

$$f(x_1+h_1,\ldots,x_n+h_n)=\sum_{|a|\le k}\frac{\partial^{|a|}f(x_1,\ldots,x_n)}{a_1!\cdots a_n!}h_1^{a_1}\cdots h_n^{a_n}+R(h_1,\ldots,h_n)$$

$|\alpha|=\alpha_1+\cdots+\alpha_n$ and $R(h_1,\ldots,h_n)$ being a residuum, which can usually be disregarded. If, for example, $c(S,X,r,T,d,\sigma)$ denotes the price of a stock option with the underlying having the price S fluctuating with a volatility of σ, the strike price being X, the riskfree interest rate for the option's time to maturity T being r and the stock yielding a dividend d, we obtain:

$$c(S+\Delta S,X+\Delta X,r+\Delta r,T+\Delta T,d+\Delta d,\sigma+\Delta\sigma)-c(S,X,r,T,d,\sigma)=$$
$$\frac{\partial c(S,X,r,T,d,\sigma)}{\partial S}\Delta S+\cdots+\frac{\partial c(S,X,r,T,d,\sigma)}{\partial\sigma}\Delta\sigma+\frac{\partial^2 c(S,X,r,T,d,\sigma)}{\partial S^2}(\Delta S)^2+\cdots+R$$

The Option Risks Regulation now assumes that in this approximation all terms except for the delta, gamma and vega term may be disregarded, which results in the following approximation of the change in the option price:

$$c(S+\Delta S,X+\Delta X,r+\Delta r,T+\Delta T,d+\Delta d,\sigma+\Delta\sigma)-c(S,X,r,T,d,\sigma)\approx$$
$$\frac{\partial c(S,X,r,T,d,\sigma)}{\partial S}\Delta S+\frac{\partial^2 c(S,X,r,T,d,\sigma)}{\partial S^2}(\Delta S)^2+\frac{\partial c(S,X,r,T,d,\sigma)}{\partial\sigma}\Delta\sigma$$

Consequently, the Option Risks Regulation is only applicable to options to which this approximation can be applied, i.e. all standard options but not exotic options, such as binary options or barrier options, the latter requiring more sophisticated procedures to assess risks and calculate the regulatory capital requirement.

The above described approximation of a function by means of a polynomial function is called a Taylor series expansion of the function in the pertinent literature. The circumstances under which a given function may be expanded into a Taylor series are described, for instance, in Heuser 1990.

2 Option Pricing Models and Sensitivities

2.1 The Black-Scholes Model for European Options

Black and Scholes were the first to show that standard put and call options can be valued by replicating them in a portfolio made up of the underlying and a cash account at a riskfree interest rate. The portfolio must be continuously adjusted to market conditions. The classical Black-Scholes model (1973) could only be used to value European put and call options on underlying instruments which do not generate a cash flow, e.g. non-dividend-paying stocks. However, the model is easy to generalize so as to permit the pricing of European options on underlying instruments which generate a cash flow, such as dividend-bearing stocks, foreign currencies and futures. In this generalized version of the model, the pricing functions to be applied to European call and put options read as follows:

$$c = e^{(b-r)T} S\, N(d_1) - e^{-rT} X\, N(d_2) \tag{2.1}$$

$$p = e^{-rT} X\, N(-d_2) - e^{(b-r)T} S\, N(-d_1) \tag{2.2}$$

with

$$d_1 = \frac{ln(S/X) + \left(b + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

c	price of the call
p	price of the put
S	current market value of the underlying
X	strike price
r	riskfree interest rate
T	time to maturity of the option in years
σ	volatility of the underlying
$N(x)$	standard normal distribution function at the point x
b	cost-of-carry of the option[7]

An option's cost-of-carry varies depending on the underlying.

[7] *As costs are unrealized profits, they are expressed in the same unit of measurement.*

2.1.1 Options on Underlying Instruments Without Cash Flows

In this case, we must set $b = r$. The formulas (2.1) and (2.2) are consequently rearranged as follows, resulting in the classical Black-Scholes pricing formula (1973):

$$c = S\,N(d_1) - e^{-rT} X\,N(d_2) \tag{2.3}$$

$$p = e^{-rT} X\,N(-d_2) - S\,N(-d_1) \tag{2.4}$$

with

$$d_1 = \frac{ln(S/X) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

This model is suitable for pricing European options on non-dividend-paying stocks and stock indices without dividends (so-called price indices).

2.1.2 Options on Stocks and Stock Indices with Known Dividend Yields

Merton (1973) extended the classical Black-Scholes formula to permit the pricing of European put and call options on stocks and stock indices for which the dividend yield q is known. In this case, the option's cost-of-carry b in the formulas (2.1) and (2.2) is given by $r - q$ and we obtain the following pricing functions:

$$c = e^{-qT} S\,N(d_1) - e^{-rT} X\,N(d_2) \tag{2.5}$$

$$p = e^{-rT} X\,N(-d_2) - e^{-qT} S\,N(-d_1) \tag{2.6}$$

with

$$d_1 = \frac{ln(S/X) + \left(r - q + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

2.1.3 Options on Foreign Currency

Garman and Kohlhagen (1983) modified the classical Black-Scholes model to cover European foreign currency options. The formula corresponds to the Merton model except that the dividend yield is replaced by a foreign currency's riskfree interest rate r_f. Hence $b = r - r_f$.

The pricing functions read:

$$c = e^{-r_f T} S\, N(d_1) - e^{-rT} X\, N(d_2) \tag{2.7}$$

$$p = e^{-rT} X\, N(-d_2) - e^{-r_f T} S\, N(-d_1) \tag{2.8}$$

with

$$d_1 = \frac{ln(S/X) + (r - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

2.1.4 Options on Futures

Black (1976) expanded the classical Black-Scholes model to make it possible to value European options on forward contracts and/or futures contracts and on bonds with current forward and futures prices F. In this case, $b = 0$ and we obtain:

$$c = e^{-rT} F\, N(d_1) - e^{-rT} X\, N(d_2) \tag{2.9}$$

$$p = e^{-rT} X\, N(-d_2) - e^{-rT} F\, N(-d_1) \tag{2.10}$$

with

$$d_1 = \frac{ln(F/X) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

2.1.5 Caps and Floors

In order to price a European cap (floor), it must first be decomposed into a portfolio of caplets (floorlets). With the model created by Black (1976), the *k*-th caplet (floorlet) can be valued as follows:

$$caplet = \frac{\tau}{1+\tau F_k} e^{-rk\tau} \left[F_k N(d_1) - X\, N(d_2) \right] \tag{2.11}$$

$$floorlet = \frac{\tau}{1+\tau F_k} e^{-rk\tau} \left[X\, N(-d_2) - F_k N(-d_1) \right] \tag{2.12}$$

with

$$d_1 = \frac{ln(F_k / X) + \left(\sigma_k{}^2 / 2\right) k\tau}{\sigma_k \sqrt{k\tau}}$$

$$d_2 = d_1 - \sigma_k \sqrt{k\tau}$$

τ denoting the interest rate period in years of the *k*-th caplet (floorlet) and F_k the forward-τ-annual interest rate p.a. of the time period $[k\tau, (k+1)\tau]$.

The above formulas are based on a face value of one. For calculating the sensitivities of a caplet (floorlet) in accordance with the Option Risks Regulation, it is the forward interest rate that constitutes the underlying (and not the forward rate agreement, which would actually be the underlying of the caplet [floorlet]), because this option falls within the risk category “interest rate options.”

2.1.6 Swaptions

A minor modification of the Black (1976) model permits to value European swaptions as follows:

$$\text{Payer swaption: } c = A\left[F\, N(d_1) - X\, N(d_2) \right] \tag{2.13}$$

$$\text{Receiver swaption: } p = A\left[X\, N(-d_2) - F\, N(-d_1) \right] \tag{2.14}$$

with

$$d_1 = \frac{ln(F/X) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$A = \frac{1}{m}\sum_{i=1}^{mn} e^{-r_i t_i}$$

n denoting the maturity of the swap expressed in years starting in T years, m the number of coupon payments per year, t_i the time to the i-th coupon date and r_i the corresponding riskfree interest rate.

The above formulas are based on a face value of one. For calculating the sensitivities of a swaption in accordance with the Option Risks Regulation, it is the forward interest rate that constitutes the underlying (and not the swap, which in reality would be the underlying of the option), because this option falls within the risk category "interest rate options."

2.2 Barone-Adesi and Whaley Approximation

By contrast to European options, American options may be exercised at any time during their time to maturity. This feature makes American options more difficult to value. With one single exception[8], we have no closed-form solutions to pricing American options. However, there are a number of analytical approximations to the valuation of American standard put and call options, such as the approximation developed by Barone-Adesi and Whaley (1987), a quadratic approximation which is easy to use and sufficiently exact for the majority of practical applications.

American Call:

$$c_{BAW} = \begin{cases} c + A_2 (S/S^*)^{q_2} & if \;\; S < S^* \\ S - X & if \;\; S \geq S^* \end{cases} \tag{2.15}$$

with

c value of the call from the respective Black-Scholes model,

[8] *The Black-Scholes model may be used to price American call options on underlying instruments without cash flow, because such options are ideally not exercised prior to their maturity.*

$$A_2 = \frac{S^*}{q_2}\left\{1 - e^{(b-r)T} N\left(d_1\left(S^*\right)\right)\right\}$$

$$d_1\left(S^*\right) = \frac{ln\left(S^*/X\right) + \left(b + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$q_2 = \frac{-\left(L-1\right) + \sqrt{\left(L-1\right)^2 + 4M/K}}{2}$$

$$M = \frac{2r}{\sigma^2}$$

$$L = \frac{2b}{\sigma^2}$$

$$K = 1 - e^{-rT}$$

The variable S* is the critical price of the underlying, above which the option should be exercised. This value is derived by solving the following non-linear equation:

$$S^* - X = c\left(S^*\right) + \left\{1 - e^{(b-r)T} N\left(d_1\left(S^*\right)\right)\right\}\frac{S^*}{q_2}$$

The equation must be solved numerically, using, for example, the Newton method. Barone-Adesi and Whaley (1987) suggest the following initial value for the iterative solution:

$$S_1^* = X + \left[S^*(\infty) - X\right]\left[1 - e^{h_2}\right]$$

with

$$h_2 = -\left(bT + 2\sigma\sqrt{T}\right)\left[\frac{X}{S^*(\infty) - X}\right]$$

$$S^*(\infty) = \frac{X}{1 - 2\left[-\left(L-1\right) + \sqrt{\left(L-1\right)^2 + 4M}\right]^{-1}}$$

American put:

$$p_{BAW} = \begin{cases} p + A_1 \left(S/S^{**}\right)^{q_1} & if \;\; S > S^{**} \\ S - X & if \;\; S \le S^{**} \end{cases} \tag{2.16}$$

with

p value of the put from the corresponding Black-Scholes model,

$$A_1 = -\frac{S^{**}}{q_1}\left\{1 - e^{(b-r)T} N\left(-d_1\left(S^{**}\right)\right)\right\}$$

$$d_1\left(S^{**}\right) = \frac{ln\left(S^{**}/X\right) + \left(b + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$q_1 = \frac{-(L-1) - \sqrt{(L-1)^2 + 4M/K}}{2}$$

$$M = \frac{2r}{\sigma^2}$$

$$L = \frac{2b}{\sigma^2}$$

$$K = 1 - e^{-rT}$$

The variable S** is the critical price of the underlying, below which the option should be exercised. This value is derived by solving the following non-linear equation:

$$X - S^{**} = p\left(S^{**}\right) - \left\{1 - e^{(b-r)T} N\left(-d_1\left(S^{**}\right)\right)\right\}\frac{S^{**}}{q_1}$$

The equation must be solved numerically, using, for example, the Newton method. Barone-Adesi and Whaley (1987) suggest the following initial value for the iterative solution:

$$S_1^{**} = S^{**}(\infty) + \left[X - S^{**}(\infty)\right]e^{h_1}$$

with

$$h_1 = \left(bT - 2\sigma\sqrt{T}\right)\left[\frac{X}{X - S^{**}(\infty)}\right]$$

$$S^{**}(\infty) = \frac{X}{1 - 2\left[-(L-1) - \sqrt{(L-1)^2 + 4M}\right]^{-1}}$$

2.3 Binomial Trees

The binomial tree method was first suggested by Cox, Ross and Rubinstein (1979) and has become one of the most frequently used numerical method for pricing American options. This procedure constructs a recombining binomial tree and is a discretization of the geometric Brownian motion, on which the continuous-time Black-Scholes model is based. The option's time to maturity is broken down into n equidistant time intervals of $\Delta t = T/n$ length. The price of the underlying can assume two different values at the end of each time step. In the Cox, Ross and Rubinstein model, the price of the underlying increases with the probability π by a fixed factor u and drops with the probability $1-\pi$ by the factor d. After j time steps, the price of the underlying can take one of the following $j+1$ values:

$$Su^i d^{j-i}, \quad i = 0,1,\ldots,j$$

with $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$ and $j = n,\ldots,0.$

The probability π of an increase in the price of the underlying is given by

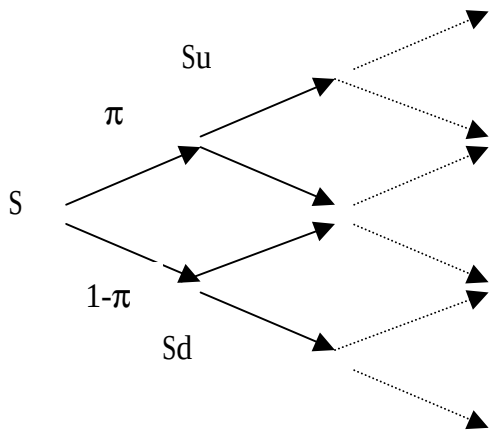
$$\pi = \frac{e^{b\Delta t} - d}{u - d}$$

with

$$b = \begin{cases} r & \textit{for options on stocks and stock indices without dividends} \\ r - q & \textit{for options on stocks and stock indices with dividends} \\ 0 & \textit{for options on forwards and futures} \\ r - r_f & \textit{for options on foreign currency} \end{cases}$$

The parameters π, u and d are chosen to ensure that the discretized version of the random variable has the same expected value and the same variance as the continuous random variable. This is to ensure that the binomial tree is the discretized version of the geometric Brownian

motion. It can be shown that for $\Delta t \to 0$ the binomial model converges towards the continuous-time model of Black and Scholes.



Graph I: Binomial Tree

The option price is calculated recursively with backward induction as follows:

Call option:

$$c_{j,i}^{CRR} = max\left(Su^i d^{j-i} - X, e^{-r\Delta t}\left[\pi\, c_{j+1,i+1}^{CRR} + (1-\pi)\, c_{j+1,i}^{CRR}\right]\right) \tag{2.17}$$

with $i = 0,1,\ldots,j;\ j = n-1,\ldots,0$ and $c_{n,i}^{CRR} = max\left(0, Su^i d^{n-i} - X\right)$ with $i = 0,1,\ldots,n$.

Put option:

$$p_{j,i}^{CRR} = max(X - Su^i d^{j-i}, e^{-r\Delta t}\left[\pi\, p_{j+1,i+1}^{CRR} + (1-\pi)\, p_{j+1,i}^{CRR}\right] \tag{2.18}$$

with $i = 0,1,\ldots,j;\ j = n-1,\ldots,0$ and $p_{n,i}^{CRR} = max\left(0, X - Su^i d^{n-i}\right)$ with $i = 0,1,\ldots,n$.

The option price error resulting from this approximation can be reduced by adjusting the price of the American option by a corrective term. If we assume that using the binomial tree, the resulting valuation error must be about the same for American and European options, the difference between the price of a European option calculated according to the Black-Scholes model and its price determined using the binomial tree method constitutes the required corrective term:

$$pr_{A,adj}^{CRR} = pr_A^{CRR} + \left(pr_E^{BS} - pr_E^{CRR}\right) \tag{2.19}$$

with

$pr_{A,adj}^{CRR}$ adjusted price of an American option valued using the binomial tree,

pr_{A}^{CRR} price of an American option valued using the binomial tree,

pr_{E}^{BS} price of the corresponding European option valued according to the Black-Scholes model,

pr_{E}^{CRR} price of the corresponding European option valued using the binomial tree:

$$pr_E^{CRR} = e^{-rt} \sum_{i=0}^{n} \binom{n}{i} \pi^i (1-\pi)^{n-i} S u^i d^{n-i}$$

This type of correction is necessary if the option price needs to be calculated as exactly as possible, as is required, for example, when calculating sensitivities (in particular sensitivities of a higher order) using numerical methods.

2.4 Sensitivities

The sensitivities of an option show how the option price changes in the event of marginal changes in certain input factors, assuming that the remaining input factors remain constant. Mathematically speaking, sensitivities are partial derivatives of the option price function with respect to the individual input factors. If sensitivities cannot be computed analytically, they are approximated by means of numerical procedures.

2.4.1 Analytical Calculation of Sensitivities

The option price formulas (2.1) and (2.2) of the generalized Black-Scholes model for pricing European options were used to calculate the following sensitivities.

Delta

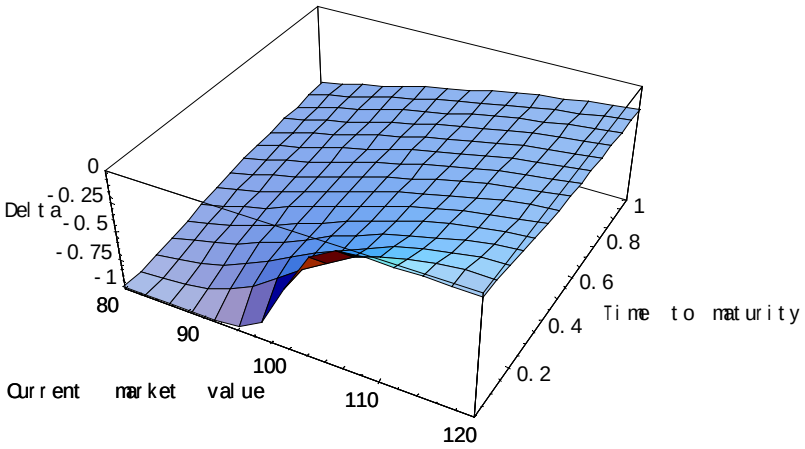
The delta of an option is defined as the change of the option price with respect to a minor change in the price of the underlying. In mathematical terms, the delta is the first partial derivative of the option price function with respect to the underlying.

Call:

$$\delta = V \frac{\partial c}{\partial S} = V e^{(b-r)T} N(d_1) \tag{2.20}$$

with *V=1* for a long position and *V=-1* for a short position.

The graph below illustrates the dependence of a call option's delta with strike 100 on the current market value of the underlying and the time to maturity.



Graph II: Delta of a call option as a function of the underlying's market price and the time to maturity

Put:

$$\delta = V\frac{\partial p}{\partial S} = Ve^{(b-r)T}\left[N(d_1)-1\right] \tag{2.21}$$

with *V=1* for a long position and *V=-1* for a short position.

The algebraic signs of delta for long and short positions in call and put options are shown in the table below:

	Long	Short
Call	+	-
Put	-	+

Table IV: Algebraic signs of delta

Gamma

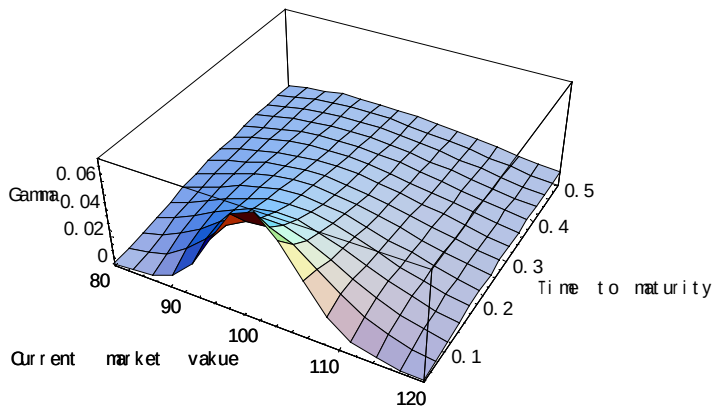
The gamma of an option shows the change of delta with respect to a minor change in the value of the underlying. In mathematical terms, the gamma is the second partial derivative of the option price function with respect to the underlying.

Call, put:

$$\gamma = V\frac{\partial^2 c}{\partial S^2} = V\frac{\partial^2 p}{\partial S^2} = V\frac{e^{(b-r)T}n(d_1)}{S\sigma\sqrt{T}} \tag{2.22}$$

with $V=1$ for a long position and $V=-1$ for a short position and $n(x)$ giving the density function of the standard normal distribution at point x.

The graph below illustrates the dependence of gamma of an option with strike 100 on the current market value of the underlying and the time to maturity.



Graph III: Gamma of an option as a function of the underlying's market price and the time to maturity

The algebraic signs of gamma for long and short positions in call and put options are shown in the table below:

	Long	Short
Call	+	-
Put	+	-

Table V: Algebraic signs of gamma

Vega

The vega of an option shows the change of the option price with respect to a minor change in the volatility of the underlying. In mathematical terms, the vega is the first partial derivative of the option price function with respect to the volatility of the underlying.

Call, put:

$$\Lambda = V\frac{\partial c}{\partial \sigma} = V\frac{\partial p}{\partial \sigma} = VSe^{(b-r)T}n(d_1)\sqrt{T} \tag{2.23}$$

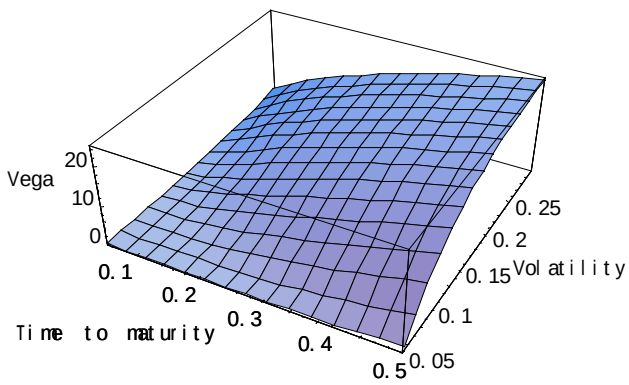
with $V=1$ for a long position and $V=-1$ for a short position.

The algebraic signs of vega for long and short positions in call and put options are shown in the table below:

	Long	Short
Call	+	-
Put	+	-

Table VI: Algebraic signs of vega

The graph below illustrates the dependence of vega on the volatility and the time to maturity.



Graph IV: Vega of an option as a function of volatility and time to maturity

Theta

The theta of an option shows the change of the option price with respect to a minor change in the time to maturity. In mathematical terms, theta is the first partial derivative of the option price function with respect to the option's time to maturity.

Call:

$$\theta = V\frac{\partial c}{\partial T} = V\left[-\frac{Se^{(b-r)T}n(d_1)\sigma}{2\sqrt{T}} - (b-r)S\,e^{(b-r)T}N(d_1) - rXe^{-rT}N(d_2)\right] \tag{2.24}$$

Put:

$$\theta = V\frac{\partial p}{\partial T} = V\left[-\frac{Se^{(b-r)T}n(d_1)\sigma}{2\sqrt{T}} + (b-r)S\,e^{(b-r)T}N(-d_1) + rXe^{-rT}N(-d_2)\right] \tag{2.25}$$

with *V=1* for a long position and *V=-1* for a short position.

The algebraic signs of theta for long and short positions in call and put options are shown in the table below:

	Long	Short
Call	**+/-**	**+/-**
Put	**+/-**	**+/-**

Table VII: Algebraic signs of theta

Rho

The rho of an option indicates the change of the option price with respect to a minor change in the riskfree interest rate. In mathematical terms, the rho is the first partial derivative of the option price function with respect to the riskfree interest rate.

Call:

$$\rho = V\frac{\partial c}{\partial r} = \begin{cases} V\,T\,Xe^{-rT}N(d_2) & for\ \ b \neq 0 \\ -V\,T\,c & for\ \ b = 0 \end{cases} \tag{2.26}$$

with *V=1* for a long position and *V=-1* for a short position.

Put:

$$\rho = V\frac{\partial p}{\partial r} = \begin{cases} -V\,T\,Xe^{-rT}N(-d_2) & for\ \ b \neq 0 \\ -V\,T\,p & for\ \ b = 0 \end{cases} \tag{2.27}$$

with *V=1* for a long position and *V=-1* for a short position.

The algebraic signs of rho for long and short positions in call and put options are shown in the table below:

	Long	**Short**
Call	**+** if $b \neq 0$, otherwise **-**	**-** if $b \neq 0$, otherwise **+**
Put	**-**	**+**

Table VIII: Algebraic signs of rho

Cost-of-carry

The sensitivity of an option's cost-of-carry shows the change of the option price with respect to a minor change in the option's cost-of-carry. In mathematical terms, the sensitivity of an option's cost-of-carry is the first partial derivative of the option price function with respect to its cost-of-carry.

Call:

$$\beta = V\frac{\partial c}{\partial b} = V\,T\,S\,e^{(b-r)T}N(d_1) \tag{2.28}$$

Put:

$$\beta = V\frac{\partial p}{\partial b} = -V\,T\,S\,e^{(b-r)T}N(-d_1) \tag{2.29}$$

with *V=1* for a long position and *V=-1* for a short position.

The algebraic signs of the sensitivity of the cost-of-carry of long and short positions in call and put options are shown in the table below:

	Long	Short
Call	+	-
Put	-	+

Table IX: Algebraic signs of the cost-of-carry

Sensitivities of caplets and floorlets

The interest-rate sensitivities of caplets/floorlets are the partial derivatives of the pricing formulas (2.11) and (2.12) with respect to the corresponding variables.

Interest-rate delta of caplets[9]:

$$\delta = V\frac{\partial c}{\partial F_k} = V\frac{\tau}{1+\tau F_k}e^{-rk\tau}N(d_1) \tag{2.30}$$

Interest-rate delta of floorlets[10]:

$$\delta = V\frac{\partial p}{\partial F_k} = V\frac{\tau}{1+\tau F_k}e^{-rk\tau}\left(N(d_1)-1\right) \tag{2.31}$$

Interest-rate gamma of caplets and floorlets:

$$\gamma = V\frac{\partial^2 c}{\partial F_k^{\,2}} = V\frac{\partial^2 p}{\partial F_k^{\,2}} = V\frac{\tau}{1+F_k}\frac{e^{-rk\tau}n(d_1)}{F_k\sigma\sqrt{k\tau}} \tag{2.32}$$

---

[9] *To compute the δ-weighted forward rate agreement (FRA), which must be decomposed into its basic components and allocated as required, it is necessary to convert the interest-rate delta into a price delta. The current market price of the FRA is given by* $S = \frac{\tau}{1+\tau F_k}e^{-rk\tau}\left[F - X\right]$. *In accordance with the chain rule for the differentiation, we obtain the following price delta:* $\frac{\partial c}{\partial S} = \frac{\frac{\partial c}{\partial F_k}}{\frac{\partial S}{\partial F_k}} = N(d_1)$.

[10] *In analogy to Footnote 9.*

Vega of caplets and floorlets:

$$\Lambda = V\frac{\partial c}{\partial \sigma} = V\frac{\partial p}{\partial \sigma} = V\frac{\tau}{1+\tau F_k}F_k e^{-rk\tau} n(d_1)\sqrt{k\tau} \tag{2.33}$$

Sensitivities of swaptions

The interest-rate sensitivities of swaptions are the partial derivatives of the price formulas (2.13) and (2.14) with respect to the corresponding variables. In the formulas below $A = \frac{1}{m}\sum_{i=1}^{mn} e^{-r_i t_i}$, n denotes the maturity of the swap expressed in years starting in T years, m the number of coupon payments per year, t_i the time to the i-th coupon date and r_i the corresponding riskfree interest rate.

Interest-rate delta of payer swaptions[11]:

$$\delta = V\frac{\partial c}{\partial F} = V \cdot A \cdot N(d_1) \tag{2.34}$$

Interest-rate delta of receiver swaptions[12]:

$$\delta = V\frac{\partial p}{\partial F} = V \cdot A \cdot (N(d_1) - 1) \tag{2.35}$$

Interest rate gamma of payer and / or receiver swaptions:

$$\gamma = V\frac{\partial^2 c}{\partial F^2} = V\frac{\partial^2 p}{\partial F^2} = V \cdot A \cdot \frac{n(d_1)}{F\sigma\sqrt{T}} \tag{2.36}$$

Vega of payer and / or receiver swaptions:

$$\Lambda = V\frac{\partial c}{\partial \sigma} = V\frac{\partial p}{\partial \sigma} = V \cdot A \cdot F \cdot n(d_1)\sqrt{T} \tag{2.37}$$

[11] *In analogy to Footnote 9.*
[12] *In analogy to Footnote 9.*

2.4.2 Numerical Calculation of Sensitivities

If sensitivities cannot be calculated analytically as is the case with the Barone-Adesi and Whaley approximation or the Cox-Ross-Rubinstein model, we have to take recourse to numerical differentiation procedures, in which the derivative is approximated by computing for difference quotients.

Delta

For computing the delta of an option using a numerical approach, we calculate two option prices. All input parameters except the price of the underlying remain unchanged. The price of the underlying is modified in both directions by a predetermined value *h*.

The numerical approximation δ_{num} of the delta of the option results as the quotient of the difference of the option prices and the difference of the corresponding values of the underlying as summarized in the following formula:

$$\delta_{num} = \frac{op(S+h) - op(S-h)}{2h} \tag{2.38}$$

The size of the assumed change *h* should be made dependent on the type of underlying:

	Stocks	Foreign currency	Interest rates	Bonds
h	1	0.01	0.0001	1

Table X: Assumed change of *h* in delta

Vega

For computing the vega of an option using a numerical approach, we calculate two option prices. All input parameters except the volatility of the underlying remain constant. The volatility of the underlying is modified in both directions by a predetermined value *h*.

The numerical approximation Λ_{num} of the vega of the option results as the quotient of the difference of the option prices and the difference of the corresponding values of the volatility of the underlying as summarized in the following formula:

$$\Lambda_{num} = \frac{op(\sigma+h) - op(\sigma-h)}{2h} \tag{2.39}$$

The size of the assumed change *h* should be made dependent on the type of underlying:

	Stocks	Foreign currency	Interest rates	Bonds
h	0.01	0.01	0.01	0.01

Table XI: Assumed change of *h* in vega

Gamma

The gamma of an option can be numerically approximated using the following difference quotient:

$$\gamma_{num} = \frac{op(S+1.5h) - op(S+0.5h) - \left(op(S-0.5h) - op(S-1.5h)\right)}{2h^2} \tag{2.40}$$

All input parameters except the value of the underlying remain unchanged.

The size of the assumed change *h* should be dependent on the type of underlying:

	Stocks	Foreign currency	Interest rates	Bonds
h	1	0.01	0.0001	1

Table XII: Assumed change of *h* in gamma

2.4.3 Annex: Numerical Differentiation

If the derivatives of a function cannot be calculated analytically, we can determine them by numerical approximation. The question regarding the size of the error resulting from numerical approximation of a derivative can only be answered correctly if the underlying function *f(x)* is differentiable a sufficient number of times. The principal approach to calculating the resulting error is illustrated below, using the example of a special case.

The first derivative of a function *f(x)* at point x_0 can be approximated numerically by

$$f'_A(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

or by

$$f'_B(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

In order to find out which of the two procedures renders more exact results, we start analyzing for the error, always provided that the function $f(x)$ is differentiable a sufficient number of times and can be expanded into a Taylor series, which at the points x_0+h and x_0-h is:

$$f(x_0+h)=f(x_0)+f'(x_0)h+f''(x_0)\frac{h^2}{2!}+f'''(x_0)\frac{h^3}{3!}+\cdots$$

or

$$f(x_0-h)=f(x_0)-f'(x_0)h+f''(x_0)\frac{h^2}{2!}-f'''(x_0)\frac{h^3}{3!}+\cdots$$

This, in turn, directly yields the following numerical derivatives:

$$f'_A(x_0)=f(x_0)+f''(x_0)\frac{h}{2!}+\cdots$$

$$f'_B(x_0)=f(x_0)+f'''(x_0)\frac{h^2}{3!}+\cdots$$

The error resulting from the first differentiation procedure is of the first order (h has the power 1), whereas the second differentiation procedure results in an error of the second order (h assumes the power 2). Hence, the second numerical differentiation procedure is definitely preferable to the first one. For further details on the error analysis of numerical differentiation procedures please refer to Schwarz (1988). It is important that the numerical procedure used to calculate derivatives is selected with care, above all, if we compute derivatives of a higher order, as otherwise the resulting derivatives may be far off the actual values.

2.5 Determining Current Volatility

The volatility of the return of the underlying is the only input parameter in the option pricing models discussed we cannot obtain directly from the market. However, it is of great importance that the volatility is calculated as exactly as possible, because this is the only input factor that contains specific information with respect to the underlying of the option. For this reason it is decisive that this parameter is derived with the utmost degree of exactness from the information available on financial markets.

2.5.1 Historical Volatility

A huge problem encountered in estimating volatility from historical data consists in determining the "correct" historical observation interval and the "correct" estimation interval. There is no optimal solution to determining either of the two. On the one hand, the data used must not be taken from an all too distant past in order to obtain the most up-to-date volatility estimate possible. On the other hand, the statistical informative value of the estimated volatility depends on the size of the sample: it is the lower, the smaller the size of the sample. In practice we generally use daily interval values (in individual cases also weekly or monthly interval values) of the price S of the underlying with a historical observation period of 250 days. As the volatility is estimated on the basis of a sample, the formula used reads as follows[13]:

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(ln\left(\frac{S_i}{S_{i-1}}\right)\right)^2 - \frac{1}{n(n-1)}\left[\sum_{i=1}^{n} ln\left(\frac{S_i}{S_{i-1}}\right)\right]^2} \qquad (2.41)$$

with

σ denoting the volatility of the underlying instrument and

S_i the *i*-th observation value of the sample chosen.

Depending on the estimation interval of the selected sample, we obtain daily, weekly or monthly volatilities. If these volatility indicators are to be rendered comparable, we have to standardize the individual results, i.e. put them on the same (annual) basis, which is done by using an annualization factor. However, this is admissible only if the individual observations are drawings of identically distributed random variables. In this case, daily volatilities are annualized by multiplying them by the factor $\sqrt{250}$, and weekly and monthly volatilities are annualized by multiplication with the factor $\sqrt{52}$ and $\sqrt{12}$, respectively. Please note in this context that option pricing models usually use annualized volatilities as input factor.

2.5.2 Implied Volatility

By contrast to historical volatility, implied volatility is computed from a single value, namely the current value of the option. For this purpose, the option's price is entered as a known input in a given option pricing model, which means that implied volatility can be calculated as the only unknown in a nonlinear equation. The option price used is the market value of the option, which results from supply and demand. A number of different numerical procedures are available for

[13] *There is a number of other procedures used to estimate historical volatility from the available data material, e.g. an exponentially weighted estimator.*

solving the nonlinear equation and for approximating the exact result as accurately as possible. Two frequently used numerical solution procedures are presented for illustration below (see Schwarz 1988).

i The Newton-Raphson Procedure

The Newton-Raphson procedure is an efficient method to calculate the implied volatility of an option. Usually only a few iterations are sufficient to approximate implied volatility with the required accuracy. In mathematical terms, the iterative procedure can be described as follows:

$$\sigma_{i+1} = \sigma_i - \frac{p(\sigma_i) - p_M}{\partial p(\sigma_i)/\partial \sigma_i} \tag{2.43}$$

with

$p(\sigma_i)$ being the option price with a volatility of σ_i and

p_M the market value of the option.

The iterative procedure is repeated until the following stop criterion is fulfilled for a predefined limit ε:

$$\left| p_M - p(\sigma_{i+1}) \right| \leq \varepsilon$$

Manaster and Koehler (1982) suggest to use the following initial value in the Newson-Raphson procedure to calculate implied volatilities using the Black-Scholes model:

$$\sigma_1 = \left[\left| ln(S/X) + rT \right| \frac{2}{T} \right]^{\frac{1}{2}}$$

ii Interval Procedure

The Newton-Raphson procedure can be used to calculate implied volatilities only if the first partial derivative of the option price with respect to the volatility can be computed analytically. If this is not possible, as for example in the case of American options, we have to take recourse to other numerical solutions. Using the interval procedure, we define an interval $[\sigma_L, \sigma_H]$ of which we can be sure that it includes the unknown implied volatility. This is the case if the market value of the option satisfies the following inequality:

$$p(\sigma_L) \leq p_M \leq p(\sigma_H)$$

The size of the initial interval is then successively reduced using the iterative procedure below until it satisfies $|p_M - p(\sigma_{i+1})| \leq \varepsilon$. In mathematical terms, the iterative procedure reads:

$$\sigma_{i+1} = \sigma_L + (p_M - p(\sigma_L))\frac{\sigma_H - \sigma_L}{p(\sigma_H) - p(\sigma_L)} \tag{2.44}$$

with σ_L having to be replaced by σ_{i+1} if $p(\sigma_{i+1}) < p_M$ and σ_H having to be replaced by σ_{i+1} in case $p(\sigma_{i+1}) > p_M$.

2.5.3 Price and Yield Volatilities for Bonds

In many cases, the volatilities of bonds are yield volatilities rather than price volatilities. These two types of volatility can be recalculated with respect to each other by using the duration. The duration D of a forward starting bond on which a bond option is based is given by:

$$D = -\frac{\sum_{i=1}^{n} t_i C_i e^{-yt_i}}{\sum_{i=1}^{n} C_i e^{-yt_i}} \tag{2.45}$$

We have the following relationship between the bond price B, its yield y and the duration D:

$$\frac{\Delta B}{B} = -Dy\frac{\Delta y}{y}$$

We thus obtain the following relationship between the price volatility σ, which enters the option pricing model used, and the yield volatility σ_y:

$$\sigma = Dy\sigma_y \tag{2.46}$$

Hence, it is always the price volatility that must be used to value bond options. If only the yield volatility is available, the price volatility must be recalculated using the above procedure.

3 Examples

The following calculations are based on the maturity-band approach. European options included in the following examples are valued using the Black-Scholes model, whereas American options are priced on the basis of binomial trees. Numerical sensitivities are calculated using the formulas (2.38) to (2.40).

3.1 Stock Options

Example 1:

We have a long position in a European call with a strike price of EUR 30, exercisable in 0.75 years on 1,000 shares quoting in EUR. The stock currently trades at EUR 32. The market data required for valuing the option are given as follows:

Riskfree interest rate: 0.03 p.a.
Volatility: 0.30
Dividend: 0.015

If we value the option using formula (2.5), we obtain a theoretical option value of EUR 4,438. According to formula (2.22), gamma is 0.0434 and vega, calculated using formula (2.23), is 10.0024.
The gamma effect, computed with formula (1.1), is EUR 142, and the vega effect, obtained with formula (1.2), is EUR 750.

Example 2:

We have a short position in an American put with a strike price of EUR 32, exercisable in 0.75 years on 1,000 shares quoting in EUR. The stock currently trades at EUR 32. The market data required for valuing the option are given as follows:

Riskfree interest rate: 0.05 p.a.
Volatility: 0.35
Dividend: 0.04

By valuing the option using formula (2.18), we obtain a theoretical option value of EUR -3,659. According to formula (2.40), gamma is -0.0408 and vega, using (2.39), is -10.6403.
The gamma effect computed with formula (1.1) is -EUR 134, and the vega effect, calculated with formula (1.2), is -EUR 931.

3.2 Stock Index Options

Example 3:

We have a short position in 5 American calls exercisable within 0.5 years on the FT-SE at strike 6000 points (1 index point corresponding to GBP 10). Currently, the index is at 6500 points. As the FT-SE is a performance index, its dividend yield must be taken into account in the option pricing model. The market data required for valuing the option are given as follows:

Riskfree interest rate: 0.053 p.a.
Volatility: 0.35
Dividend yield: 0.045
FX rate EUR/GBP: 1.4643

By valuing the option using (2.17), we obtain a theoretical option value of GBP 44,679 (corresponding to EUR 65,424). According to formula (2.40), gamma is 0.0002 and vega, using (2.39), is 1,619.5214.
The gamma effect, computed using (1.1), is GBP 1,545 (corresponding to EUR 2,262), and the vega effect, obtained with (1.2), is GBP 7,085 (corresponding to EUR 10,375).

Example 4:

We have a short position in a European put exercisable in 0.75 years on the ATX at strike 1150 points (1 index point corresponding to EUR 7.2673). The index is currently at 1100 points. Since the ATX is a price index, the dividend yield is to be set to zero. The market data required for valuing the option are given as follows:

Riskfree interest rate: 0.03 p.a.
Volatility: 0.21

Valuing the option using (2.4) yields a theoretical option value of -EUR 679. According to formula (2.22), gamma is -0.0020, and vega, computed with (2.23), is -379.8752.
The gamma effect, computed with formula (1.1), is -EUR 56, and the vega effect, obtained with formula (1.2), is -EUR 145.

3.3 FX Options

Example 5:

We have a short position in a European call with strike 118 YEN/USD exercisable in 0.0833 years on a volume of USD 1 million. The current exchange rate is 119.8903 YEN/USD. The market data required for valuing the option are given as follows:

Domestic interest rate (YEN): 0.0022 p.a.
Foreign interest rate (USD): 0.0488 p.a.
Volatility: 0.23
FX rate EUR/YEN: 0.007511

By valuing the option using (2.7), we obtain a theoretical option value of YEN 3,906,730 (corresponding to EUR 29,343). According to formula (2.22), gamma is 0.0488, and vega, obtained with (2.23), is 13.4368.
The gamma effect, computed with formula (1.1), is YEN 561,064 (corresponding to EUR 4,214), and the vega effect, calculated with (1.2), is YEN 772,615 (corresponding to EUR 5,803).

Example 6:
We have a short position in an American put exercisable within 0.5 years and with a volume of GBP 1 million at a strike price of 1.65 USD/GBP. The current exchange rate is 1.6140 USD/GBP. The market data required for valuing the option are given as follows:

Domestic interest rate (USD): 0.0490 p.a.
Foreign interest rate (GBP): 0.039 p.a.
Volatility: 0.15
FX rate EUR/USD: 0.9117

If we value the option using (2.18), we obtain a theoretical option value of USD -83,375 (corresponding to -EUR 76,013). According to formula (2.40), gamma is -2.2721 and the vega, obtained using formula (2.39), is -0.4429.
The gamma effect computed with formula (1.1) is -USD 4,735 (corresponding to -EUR 4,317) and the vega effect, obtained with formula (1.2), is -USD 16,608 (corresponding to -EUR 15,141).

3.4 Interest Rate Options

3.4.1 Bond Options

Example 7:
We have a long position in a European call at strike 100 exercisable in 1.6 years on a bond with a face value of EUR 10 million and a 5% coupon. The current market price of the bond (clean price) is 99. The coupon is paid on an annual basis, the next coupon falling due in six months. Based on the current six-month interest rate of 3% p.a. (continuously compounded), we obtain a dirty price of 99+4.93=103.93. The present value of the coupon due prior to the maturity

date of the option at the current 1.5-year annual interest rate of 3.2% p.a. (continuously compounded) and the current six-month interest rate of 3% p.a. (continuously compounded) is EUR 0.969 million. Based on the 1.6-year interest rate of 3.22% p.a. (continuously compounded), the bond has a forward value of 99.21 at the exercise date of the option[14]. This is the correct input parameter for the option pricing model if the strike price is the dirty price payable upon maturity of the option. However, if the strike price is the clean price payable upon maturity of the option, accrued interest must be taken into account in the strike price used in the formula. The market data required for valuing the option are given as follows:

Riskfree interest rate: 0.0322 p.a.
Volatility: 0.09

If we value the option using formula (2.9), we obtain a theoretical option value of EUR 392,946. According to formula (2.22), gamma is 0.0335 and vega, calculated with (2.23) is 47.5462.
The gamma effect, computed with formula (1.1), is EUR 23,216, while the vega effect, obtained with (1.2), is EUR 106,979.

Example 8:
We have a short position in an American put with strike 99 exercisable within one year on a bond with a face value of GBP 20 million and a coupon of 7%. The current forward value of the bond is calculated as described in the example above and amounts to 99.8. The market data required for valuing the option are given as follows:

Riskfree interest rate (GBP): 0.052 p.a.
Volatility: 0.11
FX rate EUR/GBP: 1.4643

Valuing the option using formula (2.18), we obtain a theoretical option value of -GBP 762,533 (corresponding to -EUR 1,116,577). According to formula (2.40), gamma is -0.0342, and vega, using (2.39), is -37.9291.
The gamma effect, computed with formula (1.1), is -GBP 35,996 (corresponding to -EUR 52,709), while the vega effect, obtained with formula (1.2), is -GBP 208,610 (corresponding to -EUR 305,467).

[14] *The forward value of a bond can also be calculated by computing the present value - at the exercise date of the option - of all payments falling due after the exercise date of the option.*

3.4.2 Options on Interest Rate Futures[15]

Example 9:

We have a long position in an American put with strike 0.043 exercisable within 0.095 years on a three-month interest rate future with a face value of GBP 1 million and a residual maturity of 0.15 years. The current forward interest rate of the future is 0.045 p.a. The market data required for valuing the option are given as follows:

Riskfree interest rate: 0.038 p.a.
Volatility: 0.18
FX rate GBP/EUR: 1.4643

If we value the option using formula (2.18), we obtain a theoretical option value of GBP 70 (corresponding to EUR 102). The interest rate gamma, according to formula (2.40), is 27.4902, and vega, obtained with formula (2.39), is 0.0009.
The gamma effect, computed with (1.1), is GBP 1,374 (corresponding to EUR 2,013), while the vega effect, obtained with formula (1.2), is GBP 43 (corresponding to EUR 63).

Example 10:

We have a short position in a European call with strike 0.05 exercisable in 0.095 years on a three-month interest rate future with a face value of GBP 1 million and a residual maturity of 0.15 years. The current forward interest rate of the future is 0.055 p.a. The market data required for valuing the option are given as follows:

Riskfree interest rate: 0.048 p.a.
Volatility: 0.18
FX rate GBP/EUR: 1.4643

If we value the option using formula (2.11), we obtain a theoretical option value of -GBP 1,240 (corresponding to -EUR 1,816). The interest rate gamma, computed by applying (2.32), is -6.9936, while vega, applying (2.33), is -0.0004.
The gamma effect, computed with formula (1.1), is -GBP 350 (corresponding to -EUR 512), and the vega effect, obtained with (1.2), is -GBP 16 (corresponding to -EUR 24).

[15] *In economic terms, an option on an interest rate future corresponds to a caplet and should therefore be treated like a caplet.*

3.4.3 Caps

Example 11:

We have a long position in a (European) cap with a residual maturity of 5 years and a face value of EUR 10 million at strike 0.055 p.a. on the six-month LIBOR as reference interest rate. The current forward interest rates for the times to maturity of the individual caplets as well as their volatilities and riskfree interest rates are shown in the table below:

	Caplet 1	Caplet 2	Caplet 3	Caplet 4	Caplet 5	Caplet 6	Caplet 7	Caplet 8	Caplet 9
Run-up period of the caplet	0.5	1	1.5	2	2.5	3	3.5	4	4.5
Forward interest rate p.a.	0.034	0.0416	0.0466	0.054	0.0601	0.0635	0.0687	0.0694	0.0736
Volatility of the forward interest rate	0.15	0.19	0.19	0.19	0.19	0.17	0.17	0.17	0.17
Riskfree interest rate for the end of the interest period p.a.	0.0344	0.0369	0.0393	0.0424	0.0453	0.0478	0.0504	0.0525	0.0546

Table XIII

The values obtained for the individual caplets based on (2.11), the interest rate gamma and the vega of each caplet obtained with the formulas (2.32) and (2.33) as well as the associated effects are shown in the table below:

	Caplet 1	Caplet 2	Caplet 3	Caplet 4	Caplet 5	Caplet 6	Caplet 7	Caplet 8	Caplet 9
Price in EUR	0	1,346	7,593	24,014	42,425	50,719	67,379	68,960	81,099
Interest rate gamma	0.0023	9.2849	14.2421	12.3419	8.7371	7.3781	5.1659	4.6376	3.5133
Vega	0	0.0031	0.0088	0.0137	0.0150	0.0152	0.0145	0.0152	0.0146
Assumed change in interest rate	0.01	0.009	0.009	0.008	0.008	0.0075	0.0075	0.0075	0.0075
Gamma effect in EUR	1	3,760	4,557	3,949	2,457	2,075	1,453	1,136	861
Vega effect in EUR	0	1,450	4,187	6,496	7,120	6,448	6,165	6,455	6,188

Table XIV

3.4.4 Floors

Example 12:

We have a long position in a floor with a residual maturity of 5 years and a face value of USD 20 million at strike 0.05 p.a. on the six-month LIBOR as reference interest rate. The current FX rate is 0.9117 EUR/USD. The current forward interest rates for the terms to maturity of the individual floorlets as well as their volatilities and riskfree interest rates are shown in the table below:

	Floorlet 1	Floorlet 2	Floorlet 3	Floorlet 4	Floorlet 5	Floorlet 6	Floorlet 7	Floorlet 8	Floorlet 9
Run-up period of the floorlet	0.5	1	1.5	2	2.5	3	3.5	4	4.5
Forward interest rate p.a.	0.034	0.0416	0.0466	0.054	0.0601	0.0635	0.0687	0.0694	0.0736
Volatility of the forward interest rate	0.15	0.19	0.19	0.19	0.19	0.17	0.17	0.17	0.17
Riskfree interest rate for the end of the interest period p.a.	0.0344	0.0369	0.0393	0.0424	0.0453	0.0478	0.0504	0.0525	0.0546

Table XV

The values obtained for the individual floorlets using formula (2.12), the interest rate gamma and the vega of each floorlet obtained with the formulas (2.32) and (2.33) as well as the associated effects are shown in the table below:

	Floorlet 1	Floorlet 2	Floorlet 3	Floorlet 4	Floorlet 5	Floorlet 6	Floorlet 7	Floorlet 8	Floorlet 9
Price in EUR	-140,963	-79,078	-53,751	-31,026	-21,540	-14,974	-11,509	-12,742	-10,897
Interest rate gamma	-0.0871	-16.3158	-16.7162	-11.3210	-7.2135	-5.7002	-3.8188	-3.5097	-2.6129
Vega	0.000	-0.0054	-0.0103	-0.0125	-0.0124	-0.0117	-0.0107	-0.0115	-0.0108
Assumed change in interest rate	0.01	0.009	0.009	0.008	0.008	0.0075	0.0075	0.0075	0.0075
Gamma effect (EUR)	-79	-12,049	-9,754	-6,606	-3,699	-2,923	-1,958	-1,568	-1,167
Vega effect (EUR)	-5	-4,646	-8,960	-10,865	-10,719	-9,084	-8,311	-8,908	-8,391

Table XVI

3.4.5 Swaptions

Example 13:

We have a long position in a European option exercisable in 3.7 years with a face value of EUR 5 million at strike 0.045 p.a. on a 10-year receiver swap. Payments fall due semi-annually. The current forward six-month interest rate is 0.043 p.a. Due to the assumed interest rate curve, the factor A in formula (2.14) takes the value 3.793. The volatility of the forward interest rate is 0.11.

If we value the option using formula (2.14), we obtain a theoretical option value of EUR 90,890. The interest rate gamma, computed by applying formula (2.36), is 165.3421, while the vega, obtained with formula (2.37), is 0.1244.
The gamma effect, computed using (1.1), is EUR 14,881, and the vega effect, obtained with formula (1.2), is EUR 17,109.

Example 14:

We have a short position in a European option exercisable in 2 years at strike 0.08 p.a. on a 5-year payer swap with a face value of EUR 20 million. Payments fall due on an annual basis. The current forward one-year interest rate is 0.0745 p.a. Due to the interest rate curve assumed, the factor A in formula (2.13) is 4.956. The volatility of the forward interest rate is 0.115.

If we value the option using formula (2.13), we obtain a theoretical option value of -EUR 270,393. The interest rate gamma, computed using formula (2.36), is -153.1272, while the vega, obtained with formula (2.37), is -0.1955.
The gamma effect, computed with formula (1.1), is -EUR 75,032, and the vega effect, obtained with formula (1.2), is -EUR 112,399.

4 Sample Portfolio for the Maturity-Band Approach

The sample portfolio is composed of the positions presented in the examples of Section 3. The following table once again provides a summary of the individual gamma and vega effects together with the respective risk categories:

	Gamma effect (EUR)	Vega effect (EUR)	Risk category
Example 1	142	750	Stocks/EUR
Example 2	-134	-931	Stocks/EUR
Example 3	2,262	10,375	Stocks/GBP
Example 4	-56	-145	Stocks/EUR
Example 5	4,214	5,803	YEN/USD
Example 6	-4,317	-15,141	USD/GBP
Example 7	23,216	106,979	MB 10/EUR
Example 8	-52,709	-305,467	MB 9/GBP
Example 9	2,013	63	MB 3/GBP
Example 10	-512	-24	MB 3/GBP
Example 11 (Caplet 1)	1	0	MB 4/EUR
Example 11 (Caplet 2)	3,760	1,450	MB 5/EUR
Example 11 (Caplet 3)	4,557	4,187	MB 6/EUR
Example 11 (Caplet 4)	3,949	6,496	MB 6/EUR
Example 11 (Caplet 5)	2,457	7,120	MB 7/EUR
Example 11 (Caplet 6)	2,075	6,448	MB 7/EUR
Example 11 (Caplet 7)	1,453	6,165	MB 8/EUR
Example 11 (Caplet 8)	1,136	6,455	MB 9/EUR
Example 11 (Caplet 9)	861	6,188	MB 9/EUR
Example 12 (Floorlet 1)	-79	-5	MB 4/USD
Example 12 (Floorlet 2)	-12,049	-4,646	MB 5/USD
Example 12 (Floorlet 3)	-9,754	-8,960	MB 6/USD
Example 12 (Floorlet 4)	-6,606	-10,865	MB 6/USD
Example 12 (Floorlet 5)	-3,699	-10,719	MB 7/USD
Example 12 (Floorlet 6)	-2,923	-9,084	MB 7/USD
Example 12 (Floorlet 7)	-1,958	-8,311	MB 8/USD
Example 12 (Floorlet 8)	-1,568	-8,908	MB 9/USD
Example 12 (Floorlet 9)	-1,167	-8,391	MB 9/USD
Example 13	14,881	17,109	MB 11/EUR
Example 14	-75,032	-112,399	MB 9/EUR

Table XVII: Gamma and vega effects of the individual positions

If we offset the individual gamma and vega effects within the risk categories against each other, we obtain the following net effects per risk category:

Risk category	Gamma effect	Vega effect
Stocks/EUR	-48	-326
Stocks/GBP	2,262	10,375
YEN/USD	4,214	5,803
USD/GBP	-4,317	-15,141
MB 3/GBP	1,501	39
MB 9/GBP	-52,709	-305,467
MB 4/EUR	1	0
MB 5/EUR	3,760	1,450
MB 6/EUR	8,506	10,683
MB 7/EUR	4,532	13,568
MB 8/EUR	1,453	6,165
MB 9/EUR	-73,035	-99,756
MB 10/EUR	23,216	106,979
MB 11/EUR	14,881	17,109
MB 4/USD	-79	-5
MB 5/USD	-12,049	-4,646
MB 6/USD	-16,360	-19,825
MB 7/USD	-6,622	-19,803
MB 8/USD	-1,958	-8,311
MB 9/USD	2,735	-17,299

Table XVIII: Net effects per risk category

For the entire portfolio we thus obtain a gamma effect of EUR 169,913 and a vega effect of EUR 662,750.

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